Problem 1

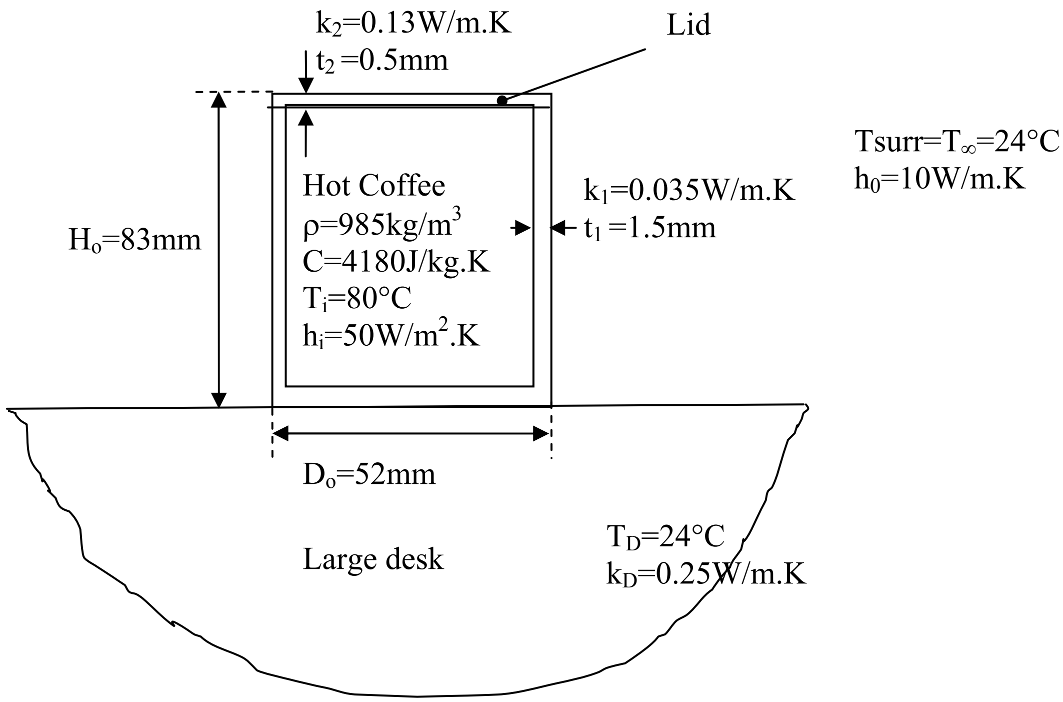
Known:

Dimensions and properties of a cup containing hot coffee as well as outside thermal conditions

Unknown:

1. Applicability of LTCM
2. Initial rate of heat loss from coffee
3. Time required to cool coffee to 60°C

Schematic



Assumptions:

1. Constant thermo-physical properties
2. Negligible conduction through the edge of the cup

Properties:

No additional properties are required

Analysis

Uniform temperature will exist if Bi<0.1, the heat is lost from coffee through Lid, Side walls and Bottom of the cup. These heat transfer processes occur in parallel to each other and each can be characterized by a different Bi#

In general

$$Bi = \frac{\text{internal resistance (heat transfer within the coffee cup)}}{\text{external resistance (from inner surface of the cup to surroundings)}}$$

i)
Consider the lid
Because of assumption 2, we ignore the edges, so that one can use internal or external dimensions of the cup or the average of them. In this solution we will use internal dimensions of the cup.

$$Bi_{\text{Heat loss through lead}} = \frac{1/h_i A_i}{\frac{t_2}{k_2 A_i} + \frac{1}{h_o A_i}} = \frac{1/h_i}{\frac{t_2}{k_2} + \frac{1}{h_o}}$$
$$= \frac{1/50}{\frac{0.0005}{0.13} + \frac{1}{10}}$$
$$= 0.196 > 0.1$$

So LTCM is not applicable for heat loss through the lid

Consider side wall

$$Bi_{\text{Heat loss t side wall}} = \frac{1/h_i A_i}{\frac{\ln(r_o/r_i)}{2\pi k_1 L_i} + \frac{1}{h_o A_i}} = \frac{1/h_i 2\pi r_i L_i}{\frac{\ln(r_o/r_i)}{2\pi k_1 L_i} + \frac{1}{h_o 2\pi r_i L_i}} = \frac{\frac{1}{50 * 0.0245}}{\frac{\ln(26/24.5)}{0.035} + \frac{1}{10 * 0.0245}}$$
$$= 0.141 > 0.1$$

Also for heat loss through the side wall LTCM is not quite applicable

Consider bottom of the cup

$$Bi_{\text{Heat loss t bottom}} = \frac{1/h_i A_i}{\frac{t_1}{k_1 A_i} + \frac{1}{k_D S}} = \frac{1/50 \cdot \pi \cdot 0.0245}{\frac{0.0015}{0.035 \cdot \pi \cdot 0.0245} + \frac{1}{0.25 \cdot 4 \cdot 0.0245}}$$
$$= 0.0063 < 0.1$$

Where S is shape factor for conduction between a disk (bottom of the cup) to semi-infinite media (case 10 in table 4.1)
LTCM is applicable

Since Bi# in two of three directions is not applicable, the LTCM for the analysis of temp of coffee as a function of time is not applicable. This means that there will be some temp gradient within the coffee at any time during the cooling process. This temp gradient will be largest close to the lid. (the largest Bi#)
In order to proceed with the solution for part 2 and 3, we assume that LTCM is applicable for part 2 and 3.

ii)

Applying energy balance on the coffee

$$\rho V c \frac{dT}{dt} = -q_{lid} - q_{side} - q_{bottom}$$

$$\frac{dT}{dt} = -\left(\frac{T - T_\infty}{R_{t,lid}} + \frac{T - T_\infty}{R_{t,side}} + \frac{T - T_D}{R_{t,bottom}}\right) \Big/ \rho V c$$

Since $T_\infty = T_D$ the above equation becomes

$$\frac{dT}{dt} = -\frac{T - T_\infty}{\rho V c}\left(\frac{1}{R_{t,lid}} + \frac{1}{R_{t,side}} + \frac{1}{R_{t,bottom}}\right)$$

Where

$$R_{t,lid} = \frac{1}{h_i \pi r_i^2} + \frac{t_2}{k_2 \pi r_i^2} + \frac{1}{h_o \pi r_i^2} = 65.68 \frac{K}{W}$$

$$R_{t,side} = \frac{1}{h_i 2\pi r_i L_i} + \frac{\ln(r_0 / r_i)}{k_1 2\pi L_i} + \frac{1}{h_o 2\pi r_i L_i} = 6.60 \frac{K}{W}$$

$$R_{t,bottom} = \frac{1}{h_i \pi r_i^2} + \frac{t_1}{k_1 \pi r_i^2} + \frac{1}{k_D S} = 74.15 \frac{K}{W}$$

At $t = 0,\ T = Ti = 80°C$,

$$\frac{dT}{dt} = -\frac{80 - 24}{985 \cdot 4180 \cdot \pi \cdot 0.0245^2 \cdot 0.081}\left(\frac{1}{65.68} + \frac{1}{6.60} + \frac{1}{74.15}\right)$$

$$= -\ 0.016 \text{ K/s}$$

iii)

Since

$$\frac{dT}{dt} = -A(T - T_\infty) \Rightarrow \frac{dT}{T - T_\infty} = -A dt \Rightarrow \ln(T - T_\infty)\Big|_{T_i}^{T} = -At \Rightarrow t = -\ln\left(\frac{T - T_\infty}{T_i - T_\infty}\right) \Big/ A$$

Where: $A = \dfrac{1}{\rho V c}\left(\dfrac{1}{R_{t,lid}} + \dfrac{1}{R_{t,side}} + \dfrac{1}{R_{t,bottom}}\right) = 2.866 \cdot 10^{-4} \dfrac{1}{s}$

$$t = -\ln\left(\frac{60 - 24}{80 - 24}\right) \Big/ 2.866 \cdot 10^{-4} = 1542 s \approx 26 \min$$

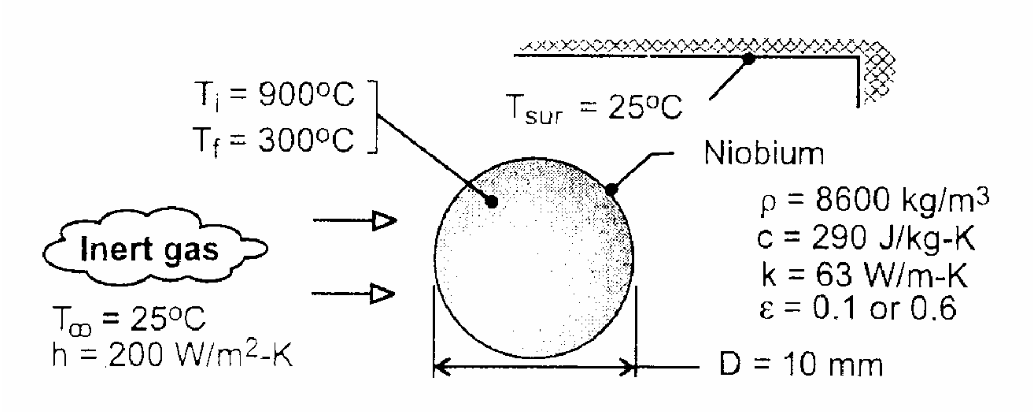
Problem 2
Known:

Initial and final temperatures of a niobium sphere
Diameter and properties of the sphere
Temperature of surroundings and gas flow and convection coefficient associated with the flow

Unknown:

1. Time required to cool the sphere exclusively by radiation
2. Time required to cool the sphere exclusively by convection
3. Combined effects of radiation and convection

Schematic:



Assumptions:

1. Uniform temperature at any time
2. Negligible effect of holding mechanism on heat transfer
3. Constant properties
4. Radiation exchange is between a small surface and large surrounding

Analysis:
If cooling is exclusively by radiation, the required time is determined from eq. (5.18) with $V = \pi D^3/6, A_{s.r} = \pi D^2$ and ε = 0.1

$$t = \frac{\rho V c}{4\varepsilon A_{s,r}\sigma T_{sur}^3}\left\{\ln\left|\frac{T_{sur}+T}{T_{sur}-T}\right| - \ln\left|\frac{T_{sur}+T_i}{T_{sur}-T_i}\right| + 2\left[\tan^{-1}\left(\frac{T}{T_{sur}}\right) - \tan^{-1}\left(\frac{T_i}{T_{sur}}\right)\right]\right\}$$

$$t = \frac{8600\cdot 290\cdot 0.01}{24\cdot(0.1)\cdot 5.67\cdot 10^{-8}\cdot 298^3}\left\{\ln\left|\frac{298+573}{298-573}\right| - \ln\left|\frac{298+1173}{298-1173}\right| + 2\left[\tan^{-1}\left(\frac{573}{298}\right) - \tan^{-1}\left(\frac{1173}{298}\right)\right]\right\}$$

$= 1190\text{s} \qquad\qquad (\varepsilon = 0.1)$

If ε = 0.6, cooling is six times faster, in which case,
t=199s

If cooling is xexlusively by convection, eq. (5.5) yields

$$t = \frac{\rho c D}{6h}\ln\left(\frac{T_i - T_\infty}{T_f - T_\infty}\right) = \frac{8600 \cdot 290 \cdot 0.01}{1200}\ln\left(\frac{875}{275}\right)$$

$$= 24.1\text{s}$$

With both radiation and convection, the temperature history may be obtained from eq. (5.15).

$$\rho(\pi D^3/6)c\frac{dT}{dt} = -\pi D^2\left[h(T - T_\infty) + \varepsilon\sigma(T^4 - T_{sur}^4)\right]$$

Rearrange above equation, yield,

$$\frac{dT}{dt} = -\,\pi D^2\left[h(T - T_\infty) + \varepsilon\sigma(T^4 - T_{sur}^4)\right]\big/\rho(\pi D^3/6)c$$

$$t = \int_{1173}^{573} \frac{\rho c(\pi D^3/6)}{\pi D^2\left[h(T - T_\infty) + \varepsilon\sigma(T^4 - T_{sur}^4)\right]}dT$$

Numerically integrating above equation from $T_i=1173$ *at* $t=0$ *to* $T=573$,

Table Numerical integration results

Number of steps (n)	**step length h**	**value of the integral**	**Error**
1	600	24.98	
2	300	20.94	4.03
3	200	20.05	0.89
4	150	19.72	0.33
5	120	19.56	0.16
6	100	19.48	0.09

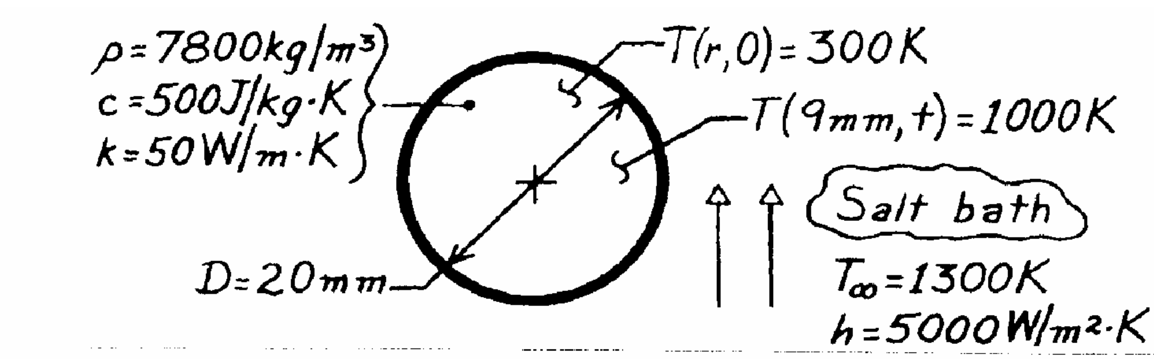
Problem 3
Known:

A ball bearing is suddenly immersed in a molten salt bath
Heat treatement to harden occurs at locations with T > 1000K

Unknown:

Time required to harden outer layer of 1mm
Total energy required to harden outer layer of 1mm

Schematic:



Assumptions:

1. One-dimensional radial conduction
2. Constant properties
3. Fo ≥0.2

Analysis:

Since any location within the ball whose temperature exceeds 1000K will be hardened the problem is to find the time when the location r=9mm reaches 1000K. then a 1mm outer layer will be hardened. Begin by finding the Biot number.

$$Bi = \frac{hr_o}{k} = \frac{5000 \cdot 0.01}{50} = 1$$

Using the one term approximate solution for a sphere, we have

$$\theta^* = \frac{T - T_\infty}{T_i - T_\infty} = C_1 e^{-\zeta_1^2 F_o} \cdot f_1$$

Where $f_1 = \dfrac{\sin(\zeta_1 r^*)}{\zeta_1 r^*}$

This lead to

$$Fo = -\frac{1}{\zeta_1^2} \ln\left[\theta^* \Big/ C_1 \frac{1}{\zeta_1 r^*} \sin(\zeta_1 r^*)\right]$$

From Table 5.1 with Bi=1, for the sphere find ζ_1=1.5708 and C_1=1.2732. with $r^* = r/r_o$ = 9/10 = 0.9, substitute numerical values.

Fo = 0.441 > 0.2

So one-term approximate is applicable
From the definition of the Fourier number with $\alpha = k/\rho c$

$$t = Fo\frac{r_o^{\ 2}\rho c}{k} = 0.441\frac{0.01^2 \cdot 7800 \cdot 500}{50} = 3.4s$$

Note that because of the properties of the ball, Fo = 0.441 corresponds to very short time.

2)
For a sphere the one-term approximate for the energy loss/gain becomes

$$\Phi = 1 - Bi\theta_0^* = 1 - \frac{3[\sin\zeta_1 - \zeta_1\cos\zeta_1]}{\zeta_1^{\ 3}}\theta_0^*$$

$$\theta_0^* = C_1 e^{-\zeta^2 Fo} = 1.2732e^{-1.5708^2 \cdot 0.441} = 0.4289$$

$$\Phi = 0.668$$

$$\Phi = Q/Q_0 \Rightarrow Q = \Phi\frac{\pi D^3 \rho c_p (T_\infty - T_i)}{6} = 10.9 \cdot 10^3 J$$

Alternatively Φ can be determined graphically using fig D9
For Bi^2*Fo = 0.441 and Bi =1 ➔ Φ =0.67
Which is very close to the value determined one-term approximation approach.