9-3 LAMINAR FLAT-PLATE BOUNDARY LAYER: EXACT SOLUTION

The solution for the laminar boundary layer on a horizontal flat plate was obtained by Prandtl's student H. Blasius [2] in 1908. For two-dimensional, steady, incompressible flow with zero pressure gradient, the governing equations of motion (Eqs. 5.27) reduce to [3]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9.3}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{9.4}$$

with boundary conditions

$$\begin{aligned} &\text{at} \quad y = 0, \quad u = 0, \quad v = 0 \\ &\text{at} \quad y = \infty, \quad u = U, \quad \frac{\partial u}{\partial y} = 0 \end{aligned} \tag{9.5}$$

Equations 9.3 and 9.4, with boundary conditions Eq. 9.5 are a set of nonlinear, coupled, partial differential equations for the unknown velocity field u and v. To solve them, Blasius reasoned that the velocity profile, u/U, should be *similar* for all values of x when plotted versus a nondimensional distance from the wall; the boundary-layer thickness, δ, was a natural choice for nondimensionalizing the distance from the wall. Thus the solution is of the form

$$\frac{u}{U} = g(\eta) \quad \text{where} \quad \eta \propto \frac{y}{\delta} \tag{9.6}$$

Based on the solution of Stokes [4], Blasius reasoned that $\delta \propto \sqrt{\nu x/U}$ and set

$$\eta = y\sqrt{\frac{U}{\nu x}} \tag{9.7}$$

We now introduce the stream function, ψ, where

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \tag{5.4}$$

satisfies the continuity equation (Eq. 9.3) identically. Substituting for u and v into Eq. 9.4 reduces the equation to one in which ψ is the single dependent variable. Defining a dimensionless stream function as

$$f(\eta) = \frac{\psi}{\sqrt{\nu x U}} \tag{9.8}$$

makes $f(\eta)$ the dependent variable and η the independent variable in Eq. 9.4. With ψ defined by Eq. 9.8 and η defined by Eq. 9.7, we can evaluate each of the terms in Eq. 9.4.

The velocity components are given by

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = \sqrt{\nu x U}\,\frac{df}{d\eta}\sqrt{\frac{U}{\nu x}} = U\frac{df}{d\eta} \tag{9.9}$$

and

$$v = -\frac{\partial \psi}{\partial x} = -\left[\sqrt{\nu x U}\,\frac{\partial f}{\partial x} + \frac{1}{2}\sqrt{\frac{\nu U}{x}}\,f\right] = -\left[\sqrt{\nu x U}\,\frac{df}{d\eta}\left(-\frac{1}{2}\,\eta\,\frac{1}{x}\right) + \frac{1}{2}\sqrt{\frac{\nu U}{x}}\,f\right]$$

or

$$v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\left[\eta\,\frac{df}{d\eta} - f\right] \tag{9.10}$$

By differentiating the velocity components, it also can be shown that

$$\frac{\partial u}{\partial x} = -\frac{U}{2x}\,\eta\,\frac{d^2 f}{d\eta^2}$$

$$\frac{\partial u}{\partial y} = U\sqrt{U/\nu x}\,\frac{d^2 f}{d\eta^2}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{U^2}{\nu x}\,\frac{d^3 f}{d\eta^3}$$

Substituting these expressions into Eq. 9.4, we obtain

$$2\,\frac{d^3 f}{d\eta^3} + f\,\frac{d^2 f}{d\eta^2} = 0 \tag{9.11}$$

with boundary conditions:

$$\text{at} \quad \eta = 0, \quad f = \frac{df}{d\eta} = 0$$

$$\text{at} \quad \eta \to \infty, \quad \frac{df}{d\eta} = 1 \tag{9.12}$$

The second-order partial differential equations governing the growth of the laminar boundary layer on a flat plate (Eqs. 9.3 and 9.4) have been transformed to a nonlinear, third-order ordinary differential equation (Eq. 9.11) with boundary conditions given by Eq. 9.12. It is not possible to solve Eq. 9.11 in closed form; Blasius solved it using a power series expansion about $\eta = 0$ matched to an asymptotic expansion for $\eta \to \infty$. The same equation later was solved more precisely—again using numerical methods—by Howarth [5], who reported results to 5 decimal places. The numerical values of f, $df/d\eta$, and $d^2f/d\eta^2$ in Table 9.1 were calculated with a personal computer using 4th-order Runge-Kutta numerical integration.

The velocity profile is obtained in dimensionless form by plotting u/U versus η, using values from Table 9.1. The resulting profile is sketched in Fig. 9.3*b*. Velocity profiles measured experimentally are in excellent agreement with the analytical solution. Profiles from all locations on a flat plate are similar; they collapse to a single profile when plotted in nondimensional coordinates.

From Table 9.1, we see that at $\eta = 5.0$, $u/U = 0.992$. With the boundary-layer thickness, δ, defined as the value of y for which $u/U = 0.99$, Eq. 9.7 gives

$$\delta \approx \frac{5.0}{\sqrt{U/\nu x}} = \frac{5.0x}{\sqrt{Re_x}} \tag{9.13}$$

Table 9.1 The Function $f(\eta)$ for the Laminar Boundary Layer along a Flat Plate at Zero Incidence

$\eta = y\sqrt{\dfrac{U}{\nu x}}$	f	$f' = \dfrac{u}{U}$	f''
0	0	0	0.3321
0.5	0.0415	0.1659	0.3309
1.0	0.1656	0.3298	0.3230
1.5	0.3701	0.4868	0.3026
2.0	0.6500	0.6298	0.2668
2.5	0.9963	0.7513	0.2174
3.0	1.3968	0.8460	0.1614
3.5	1.8377	0.9130	0.1078
4.0	2.3057	0.9555	0.0642
4.5	2.7901	0.9795	0.0340
5.0	3.2833	0.9915	0.0159
5.5	3.7806	0.9969	0.0066
6.0	4.2796	0.9990	0.0024
6.5	4.7793	0.9997	0.0008
7.0	5.2792	0.9999	0.0002
7.5	5.7792	1.0000	0.0001
8.0	6.2792	1.0000	0.0000

The wall shear stress may be expressed as

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right]_{y=0} = \mu U \sqrt{U/\nu x} \left.\frac{d^2 f}{d\eta^2}\right]_{\eta=0}$$

Then

$$\tau_w = 0.332U\sqrt{\rho\mu\, U/x} = \frac{0.332\rho U^2}{\sqrt{Re_x}} \tag{9.14}$$

and the wall shear stress coefficient, C_f, is given by

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}} \tag{9.15}$$

Each of the results for boundary-layer thickness, δ, wall shear stress, τ_w, and skin friction coefficient, C_f, Eqs. 9.13 through 9.15, depends on the length Reynolds number, Re_x, to the one-half power. The boundary-layer thickness increases as $x^{1/2}$, and the wall shear stress and skin friction coefficient vary as $1/x^{1/2}$. These results characterize the behavior of the laminar boundary layer on a flat plate.

EXAMPLE 9.2 Laminar Boundary Layer on a Flat Plate: Exact Solution

Use the numerical results presented in Table 9.1 to evaluate the following quantities for laminar boundary-layer flow on a flat plate:

(a) δ^*/δ (for $\eta = 5$ and as $\eta \to \infty$).
(b) v/U at the boundary-layer edge.
(c) Ratio of the slope of a streamline at the boundary-layer edge to the slope of δ versus x.

EXAMPLE PROBLEM 9.2

GIVEN: Numerical solution for laminar flat-plate boundary layer, Table 9.1.

FIND: **(a)** δ^*/δ (for $\eta = 5$ and as $\eta \to \infty$).
(b) v/U at boundary-layer edge.
(c) Ratio of the slope of a streamline at the boundary-layer edge to the slope of δ versus x.

SOLUTION:
The displacement thickness is defined by Eq. 9.1 as

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \approx \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

In order to use the Blasius exact solution to evaluate this integral, we need to convert it from one involving u and y to one involving f' $(= u/U)$ and η variables. From Eq. 9.7, $\eta = y\sqrt{\frac{U}{\nu x}}$, so $y = \eta\sqrt{\frac{\nu x}{U}}$ and $dy = d\eta\sqrt{\frac{\nu x}{U}}$

Thus,

$$\delta^* = \int_0^{\eta_{max}} (1 - f')\sqrt{\frac{\nu x}{U}}\, d\eta = \sqrt{\frac{\nu x}{U}} \int_0^{\eta_{max}} (1 - f')\, d\eta \qquad (1)$$

Note: Corresponding to the upper limit on y in Eq. 9.1, $\eta_{max} = \infty$, or $\eta_{max} \approx 5$.
From Eq. 9.13,

$$\delta \approx \frac{5}{\sqrt{U/\nu x}}$$

so if we divide each side of Eq. 1 by each side of Eq. 9.13, we obtain (with $f' = df/d\eta$)

$$\frac{\delta^*}{\delta} = \frac{1}{5}\int_0^{\eta_{max}} \left(1 - \frac{df}{d\eta}\right) d\eta$$

Integrating gives

$$\frac{\delta^*}{\delta} = \frac{1}{5}[\eta - f(\eta)]_0^{\eta_{max}}$$

Evaluating at $\eta_{max} = 5$, we obtain

$$\frac{\delta^*}{\delta} = \frac{1}{5}(5.0 - 3.2833) = 0.343$$

$\frac{\delta^*}{\delta}(\eta = 5)$

The quantity $\eta - f(\eta)$ becomes constant for $\eta > 7$. Evaluating at $\eta_{max} = 8$ gives

$$\frac{\delta^*}{\delta} = \frac{1}{5}(8.0 - 6.2792) = 0.344$$

$\frac{\delta^*}{\delta}(\eta \to \infty)$

Thus, $\delta^*_{\eta\to\infty}$ is 0.24 percent larger than $\delta^*_{\eta=5}$.
From Eq. 9.10,

$$v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\left(\eta\frac{df}{d\eta} - f\right), \quad \text{so} \quad \frac{v}{U} = \frac{1}{2}\sqrt{\frac{\nu}{Ux}}\left(\eta\frac{df}{d\eta} - f\right) = \frac{1}{2\sqrt{Re_x}}\left(\eta\frac{df}{d\eta} - f\right)$$

Evaluating at the boundary-layer edge ($\eta = 5$), we obtain

$$\frac{v}{U} = \frac{1}{2\sqrt{Re_x}}[5(0.9915) - 3.2833] = \frac{0.837}{\sqrt{Re_x}} \approx \frac{0.84}{\sqrt{Re_x}} \longleftarrow \frac{v}{U}(\eta = 5)$$

Thus v is only 0.84 percent of U at $Re_x = 10^4$, and only about 0.12 percent of U at $Re_x = 5 \times 10^5$.

The slope of a streamline at the boundary-layer edge is

$$\left.\frac{dy}{dx}\right)_{\text{streamline}} = \frac{v}{u} = \frac{v}{U} \approx \frac{0.84}{\sqrt{Re_x}}$$

The slope of the boundary-layer edge may be obtained from Eq. 9.13,

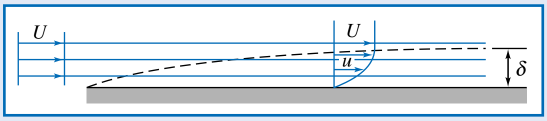
$$\delta \approx \frac{5}{\sqrt{U/\nu x}} = 5\sqrt{\frac{\nu x}{U}}$$

so

$$\frac{d\delta}{dx} = 5\sqrt{\frac{\nu}{U}}\,\frac{1}{2}x^{-1/2} = 2.5\sqrt{\frac{\nu}{Ux}} = \frac{2.5}{\sqrt{Re_x}}$$

$$\text{Thus } \left.\frac{dy}{dx}\right)_{\text{streamline}} = \frac{0.84}{2.5}\frac{d\delta}{dx} = 0.336\frac{d\delta}{dx} \longleftarrow \left.\frac{dy}{dx}\right)_{\text{streamline}}$$

This result indicates that the slope of the streamlines is about 1/3 of the slope of the boundary layer edge—the streamlines penetrate the boundary layer, as sketched below:



This problem illustrates use of numerical data from the Blasius solution to obtain other information on a flat plate laminar boundary layer, including the result that the edge of the boundary layer is *not* a streamline.