Numerical integration using Simpson's Rule

A curve $y=f(x)$ can be integrated using Simpson's Rule, as follows:

$$\int_a^b y\,dx = \frac{(b-a)}{3n}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + y_n\right]$$

$$= \frac{(b-a)}{3n}\left[y_0 + y_n + 4\sum_{m=1}^{n-1} y_{m_{odd}} + 2\sum_{m=1}^{n-2} y_{m_{even}}\right]$$

Where n is the number of equal steps (not points). Note that n must be an even number.

Table 9.1 The Function $f(\eta)$ for the Laminar Boundary Layer along a Flat Plate at Zero Incidence

$\eta = y\sqrt{\frac{U}{\nu x}}$	f	$f' = \frac{u}{U}$	f''
0	0	0	0.3321
0.5	0.0415	0.1659	0.3309
1.0	0.1656	0.3298	0.3230
1.5	0.3701	0.4868	0.3026
2.0	0.6500	0.6298	0.2668
2.5	0.9963	0.7513	0.2174
3.0	1.3968	0.8460	0.1614
3.5	1.8377	0.9130	0.1078
4.0	2.3057	0.9555	0.0642
4.5	2.7901	0.9795	0.0340
5.0	3.2833	0.9915	0.0159
5.5	3.7806	0.9969	0.0066
6.0	4.2796	0.9990	0.0024
6.5	4.7793	0.9997	0.0008
7.0	5.2792	0.9999	0.0002
7.5	5.7792	1.0000	0.0001
8.0	6.2792	1.0000	0.0000

Example: Blasius boundary layer calculations (displacement and momentum thicknesses).

$$\delta^* = \int_0^\delta \left(1-\frac{u}{U}\right)dy$$

$$\delta^* = \frac{\delta}{5}\int_0^5 (1-f')\,d\eta$$

$$\delta^* = \frac{\delta}{5}[\eta]_0^5 - \frac{\delta}{5}\int_0^5 f'\,d\eta$$

$$\delta^* = \delta - \frac{\delta}{5}\int_0^5 f'\,d\eta$$

$$\vartheta = \int_0^\delta \left(\frac{u}{U} - \left(\frac{u}{U}\right)^2\right)dy$$

$$\vartheta = \frac{\delta}{5}\int_0^5 \left(f' - (f')^2\right)d\eta$$

$$\theta = \frac{\delta}{5}\left(\int_0^5 f'\,d\eta - \int_0^5 (f')^2\,d\eta\right)$$

$$Note: \frac{\eta}{y} = \sqrt{\frac{U}{\nu x}} \quad (Table\ 9.1)$$

$$also: \frac{5}{\delta} = \sqrt{\frac{U}{\nu x}} \quad (Eq.\ 9.13)$$

$$so\ \ y = \frac{\delta\eta}{5}\ and\ dy = \frac{\delta}{5}d\eta$$

Evaluate the integrals using Simpson's Rule.

In this case, a=0, b=5, and use the ten steps from $0 < \eta < 5.0$. These are evenly spaced with $\Delta\eta$=0.5 and n=10, yielding:

$$\int_0^5 y\,dx = \frac{(5-0)}{3\cdot 10}\left[y_0 + y_{10} + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

For the displacement thickness, δ^*, $y=f'$, listed in Table 9.1, above.

$$\int_0^5 f'\,d\eta = \frac{0.5}{3}\left[0 + 0.9915 + 4(0.1659 + 0.4868 + 0.7513 + 0.9130 + 0.9795) + 2(0.3298 + 0.6298 + 0.8460 + 0.9555)\right]$$

$$\int_0^5 f'\,d\eta = 3.283$$

The momentum thickness, θ, requires the above value, and another integral, where $y=(f')^2$, which is also listed in the table above.

$$\int_0^5 (f')^2\, d\eta = \frac{0.5}{3}\left[0+0.9915^2+4\left(0.1659^2+0.4868^2+0.7513^2+0.9130^2+0.9795^2\right)+2\left(0.3298^2+0.6298^2+0.8460^2+0.9555^2\right)\right]$$

$$\int_0^5 (f')^2\, d\eta = 2.623$$

$$\delta^* = \delta - \frac{\delta}{5}(3.283) = 0.343\delta$$

$$\frac{\delta^*}{\delta} = 0.343 \text{ \textit{(compare with 0.344 in textbook)}}$$

$$\theta = \frac{\delta}{5}(3.283 - 2.623) = 0.132\delta$$

$$\frac{\vartheta}{\delta} = 0.132 \text{ \textit{(compare with 0.133 in textbook)}}$$

Errors in the third decimal place can be attributed to rounding off in the calculations.