THE VORTEX PANEL METHOD

a) Approximate the contour of the airfoil by an inscribed polygon with \( m \) sides, called “panels”. Number the panels clockwise with panel \#1 starting on the lower (pressure) side, at the trailing edge and panel \#\( m \) ending on the trailing edge from the upper (suction) side.

b) Assume that each panel represents a planar vortex sheet with linearly varying strength and such that the end strength of each panel is the same as the starting strength of the next panel:

\[
\gamma(s_j) = \gamma_j + \frac{s_j}{S_j} (\gamma_{j+1} - \gamma_j)
\]

Exception: \( \gamma_1 \neq \gamma_{m+1} \)
Instead of the dimensional strength \( \gamma \) (in velocity units), it is more convenient to use the dimensionless strength \( \gamma' \), defined as

\[
\gamma' = \frac{\gamma}{2\pi V_\infty}
\]

Then,

\[
\gamma'(s_j) = \gamma'_j + \frac{s_j}{S_j} (\gamma'_{j+1} - \gamma'_j)
\]

c) The only unknowns of this problem are the end strengths \( \gamma'_j \); \( j = 1, 2, ..., m + 1 \). They can be found by solving \( m + 1 \) equations consisting of

i) \( m \)-equations, expressing the no penetration condition at the \( m \) mid-points of the panels (called “control points”)

ii) one equation, expressing the Kutta condition.

Once \( \gamma_j \) are found, the velocity can be found by superposition and the pressure can be computed from Bernoulli’s equation.

d) Apply the Kutta condition at the trailing edge. Note that strict application of K-c would require that

\[
(V_u)_{t.e.} = (V_l)_{t.e.} = 0
\]

Instead, apply the weaker condition

\[
(V_u - V_l)_{t.e.} = 0
\]

which implies that

\[
\gamma'_1 + \gamma'_{m+1} = 0 \quad \text{(A)}
\]

e) Apply the no penetration condition at the \( m \) control points (mid-panels). Notice that all panels are assumed to be parts of the same streamline. The equipotential lines are normal to streamlines and, therefore, to panels. Let \( \phi \) be the velocity potential. Then,

\[
\nabla \phi = \vec{V}, \text{ which implies that } V_{ni} = \frac{\partial \phi}{\partial n_i}
\]
where $V_{ni}$ is the velocity component normal to the panel $i$, and $\partial \phi / \partial n_i$ is the derivative of $\phi$ in the direction of the normal vector, $\vec{n}_i$. Therefore, the $n$-equation of no penetration becomes

$$\frac{\partial \phi(x_i, y_i)}{\partial n_i} = 0, \quad i = 1, 2, \ldots, m \quad (B)$$

f) The velocity potential $\phi(x_i, y_i)$ at a control point can be found by superposition of the potential due to the uniform stream and the potentials due to the $m$ vortex panels, as

$$\phi(x_i, y_i) = V_{\infty}(x_i \cos \alpha + y_i \sin \alpha) + \sum_{j=1}^{m} \int_{\text{panel } j} \frac{-\gamma(s_j)}{2\pi} \tan^{-1} \left( \frac{y_i - y_j}{x_i - x_j} \right) ds_j \quad (C)$$

The remaining work is only algebraic and trigonometric manipulations.
g) Notice that
\[ \frac{\partial x_i}{\partial n_i} \approx \frac{\delta x_i}{\delta n_i} = \sin \theta_i \]
\[ \frac{\partial y_i}{\partial n_i} \approx \frac{\delta y_i}{\delta n_i} = \cos \theta_i \]

Then, by chain rule,
\[ \frac{\partial \phi}{\partial n_i} = \frac{\partial \phi}{\partial x_i} \frac{\partial x_i}{\partial n_i} + \frac{\partial \phi}{\partial y_i} \frac{\partial y_i}{\partial n_i} = \frac{\partial \phi}{\partial x_i} \sin \theta_i + \frac{\partial \phi}{\partial y_i} \cos \theta_i \]

In expression (C), everything has already been expressed in terms of \( x_i, y_i \), so it is straightforward to apply the chain rule to the various derivatives.

Reminder
\[ \frac{\partial}{\partial t} \tan^{-1}[f(t)] = \frac{1}{1 + f^2(t)} \frac{df}{dt} \]

Then
\[ \frac{\partial}{\partial x_i} \left[ \tan^{-1} \left( \frac{y_i - y_j}{x_i - x_j} \right) \right] = \frac{-1}{1 + \left( \frac{y_i - y_j}{x_i - x_j} \right)^2} \frac{y_i - y_j}{(x_i - x_j)^2} \]
\[ \frac{\partial}{\partial y_i} \left[ \tan^{-1} \left( \frac{y_i - y_j}{x_i - x_j} \right) \right] = \frac{1}{1 + \left( \frac{y_i - y_j}{x_i - x_j} \right)^2} \frac{1}{x_i - x_j} \]

h) The integral in eq. (C) is w.r.t. the \( j \) panel, therefore \( x_i, y_i \) are constant during integration. Because \( x_j, y_j, s_j \) vary, it is necessary to express all of them in terms of one variable, e.g. \( s_j \).
\[ x_j = X_j - s_j \cos \theta_j \]
\[ y_j = Y_j + s_j \sin \theta_j \]

Also \( \gamma(s_j) \) has been expressed in terms of \( s_j \). Then the integral for panel \( j \) becomes
\[
\gamma'_j \int_0^{S_j} f_{1j}(s_j)\,ds_j + \gamma'_{j+1} \int_0^{S_j} f_{2j}(s_j)\,ds_j
\]
which can be computed numerically.

i) Following various manipulations (see textbook for some details) one gets the system of \( m + 1 \) linear, algebraic equations
\[
\sum A_{nij} \gamma'_j = \sin(\theta_i - \alpha), \quad i = 1, 2, \ldots, m
\]
\[
\gamma'_1 + \gamma'_{m+1} = 0
\]
where the constants \( A_{nij} \) depend only on the coordinates of the end points of panels \( i \) and \( j \) and can be computed immediately once the panels have been selected. Notice that the effect of panel \( i \) on itself has also been taken into account. The system of the above equations is solved numerically to provide the end strengths.
\[
\gamma'_j, \quad j = 1, 2, \ldots, m + 1
\]

j) The **velocity at the control points is tangential** to the panel and can be found as
\[
V_i = V_{ti} = \frac{\partial \phi}{\partial t_i} = \frac{\partial \phi}{\partial x_i} \frac{\partial x_i}{\partial t_i} + \frac{\partial \phi}{\partial y_i} \frac{\partial y_i}{\partial t_i}
\]
or
\[
V_i = \frac{\partial \phi}{\partial x_i} \cos \theta_i + \frac{\partial \phi}{\partial y_i} \sin \theta_i
\]
Instead of the dimensional velocity $V_i$, the program computes a dimensionless velocity as

$$\forall_i = \frac{V_i}{V_\infty} = \frac{1}{V_\infty} \frac{\partial \phi}{\partial t_i}$$

Following various manipulations, this is found as

$$\forall_i = \sum_{j=1}^{m+1} A_{ij} \gamma_j' + \cos(\theta_i - \alpha), \quad i = 1, 2, \ldots, m$$

where the coefficients $A_{ij}$ depend on the end coordinates of panels $i$ and $j$ alone.

k) The pressure at the control points can be found from Bernoulli’s eq. as

$$C_{P_i} = \frac{P_i - P_\infty}{\frac{1}{2} \rho V_\infty^2} = 1 - \forall_i^2$$

l) The lift can be found in two ways:

i) By integrating all vortex strengths and using the Kutta-Joukowsky theorem.

ii) By integrating the surface pressure.