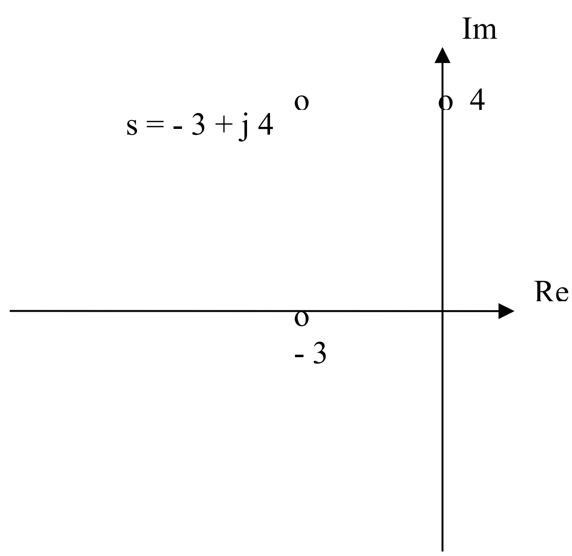
2. The Laplace Transform

2.1 Review of Laplace Transform Theory

Pierre Simon Marquis de **Laplace** (1749-1827) –
French astronomer, mathematician and politician,
Minister of Interior for 6 weeks under Napoleon,
President of Academie Francaise under Louis XVI.

Complex Variable = variable consisting of real and imaginary quantities **$s = \sigma + j\,\omega$**

Graphical Representation of Complex Numbers in complex plane (Re = real , Im = imaginary) or (σ , ω)
Example, $s = \sigma + j\,\omega = -3 + j\,4$



Complex Function

Functions of complex numbers are called complex functions. As an example:

$$G(s) = G_x + j\ G_y$$

Both G_x and G_y are real quantities.

The magnitude of G(s) is

$$|G(s)| = \sqrt{(G_x^2 + G_y^2)}$$

The angle θ of G(s) is

$$\theta = \tan^{-1} (G_y / G_x)$$

The angle takes positive values when measured counterclockwise from the positive real axis.

The complex conjugate of the function G(s) is

$$G(s)^* = G_x - j\ G_y$$

Poles and Zeros

poles = s-values for which the function G(s) tends toward infinity

zeros = s-values for which the function G(s) equals zero

Example

The complex function

$$G(s) = \frac{K(s+1)(s+15)}{s(s+2)(s+5)(s+25)^2}$$

has

- zeros at s = -1, -15
- single poles at s = 0, -2, -5,
- a double pole (multiple pole of order 2) at s = -25.

Note that G(s) → 0 as s →∞, i.e
s →∞ is an infinite zero
while
s = -2, -10 are finite zeros.

Example

The above complex function is equivalent to

$$G(s) = \frac{Ks^2(1+1/s)(1+15/s)}{s^5(1+2/s)(1+5/s)(1+25/s)} = \frac{K}{s^3} \cdot \frac{(1+1/s)(1+15/s)}{(1+2/s)(1+5/s)(1+25/s)}$$

For large values of s

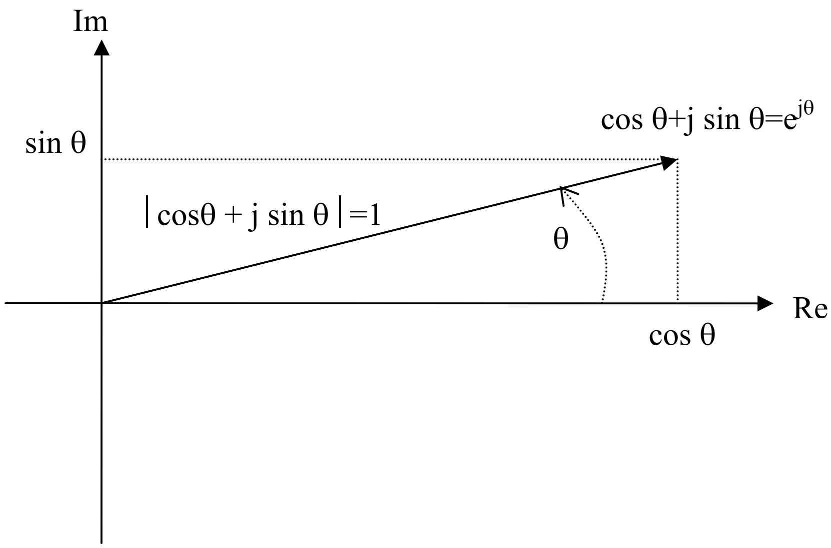
$$\lim_{s\to\infty} G(s) \approx \frac{K}{s^3}$$

i.e. G(s) has a triple zero at s → ∞.

If infinite zeros are included, G(s) has the same number of poles and zeros, 5 poles and 5 zeros.

Euler Theorem

$$\cos\theta + j\sin\theta = e^{j\theta}$$



From this results

$$\cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin\theta = e^{-j\theta}$$

and

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

2.2 Review of Laplace Transform

The **Laplace transform** is used to
- transform systems of differential equations (of real independent variable, time) into sets of algebraic equations (of complex variable, s).
- obtain easily solutions of the sets of algebraic equations.
- obtain solutions of the original problems as functions of time, applying the **Inverse Laplace transform,**

Definitions:

$f(t)$ = a function of independent variable time t, defined as non-zero for $t \geq 0$ i.e.

$f(t) = 0$ for $t<0$

F(s) = Laplace transform of $f(t)$;

Laplace transform of transforms f(t) from the t-space into F(s) in the s-space"

s = a complex variable

L = an operator indicating that the quantity that it prefixes is to be transformed by the Laplace integral $\int_0^\infty f(t)e^{-st}dt$, i.e

$$F(s) = L\{f(t)\}$$

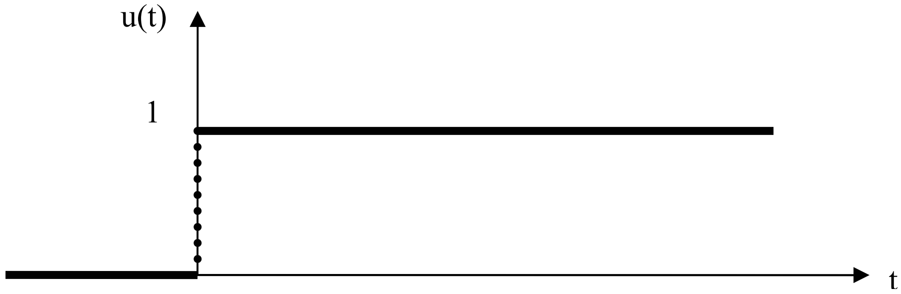
The Laplace transform of a function *f*(t) is given by

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

L[*f*(t)] is an operator applied to f(t) that does the following:

- multiplies $f(t)$ with e^{-st}
-integrates with regard to time t the product between 0 and ∞: $\int_0^\infty f(t)e^{-st}dt$ and returns a complex function F(s).

Example: Unit step function u(t)



u(t) = 0 for t < 0

1 for t > 0

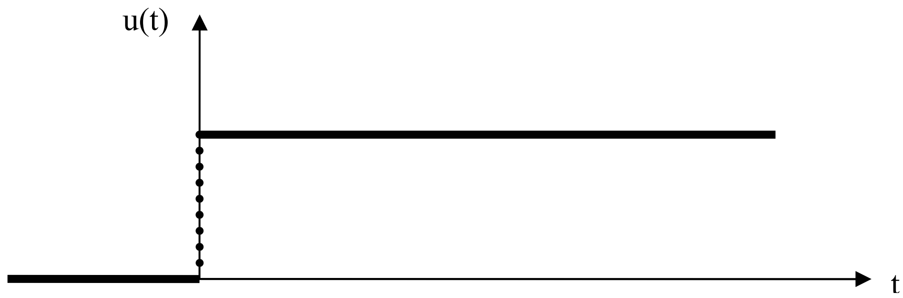
undefined for t = 0, i.e. can take any value between 0 and 1.

$$U(s) = L\{u(t)\} = \int_0^\infty 1 \cdot e^{-st} dt = \frac{1}{s}$$

U(s) has no finite zero and one zero value pole.

s-plane representation of zeros and poles results in one pole marked x in the origin

Example: Unit ramp function



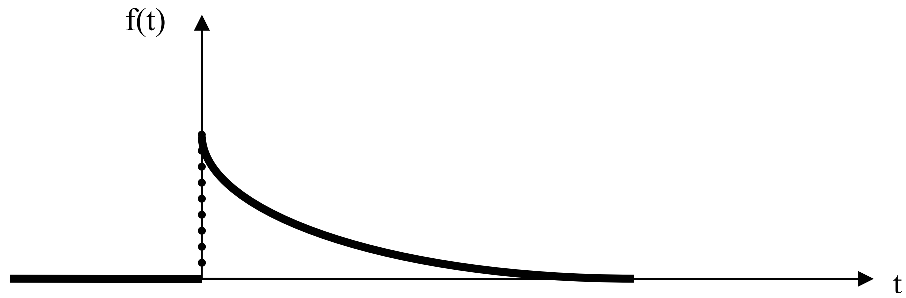
f(t) = 0 for t ≤ 0
t for t > 0

$$F(s) = L\{f(t)\} = \int_0^\infty t \cdot e^{-st} dt = \frac{1}{s^2}$$

U(s) has no finite zero and two zero value pole xx.

s-plane representation of zeros and poles results in two poles marked xx in the origin

Example: Exponential function

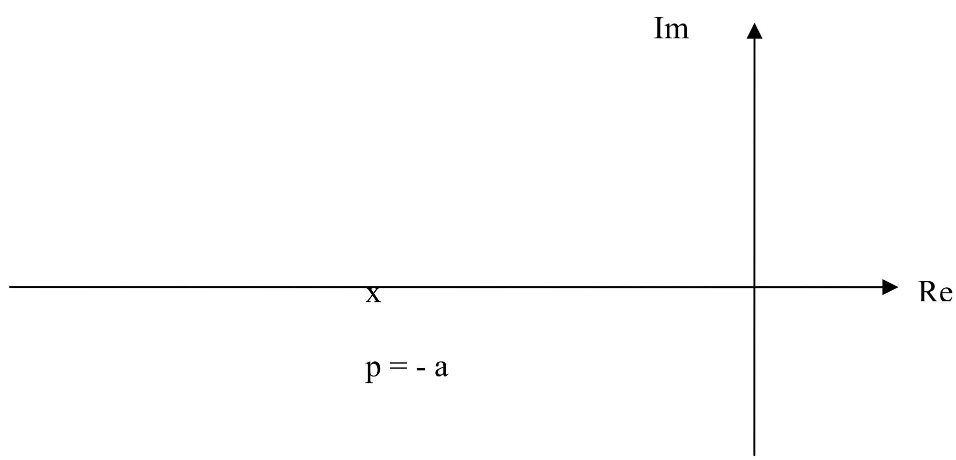


$f(t) = 0$ for $t \leq 0$
e^{-at} for $t > 0$

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-at} \cdot e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt = \frac{1}{s+a}$$

F(s) has no finite zero and one non-zero pole p = - a .

s-plane representation of zeros and poles results in pole -a marked x



Example: Sinusoidal function

f(t) = 0 for $t \leq 0$
sin (ωt) for t > 0

where, Euler identity gives

$$\sin \omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

$$F(s) = L\{f(t)\} = \int_0^{\infty} \sin(\omega t) \cdot e^{-st} dt = \frac{1}{2j}\int_0^{\infty} (e^{-(j\omega)t} + e^{-(j\omega)t}) \cdot e^{-st} dt = \frac{\omega}{s^2 + \omega^2}$$

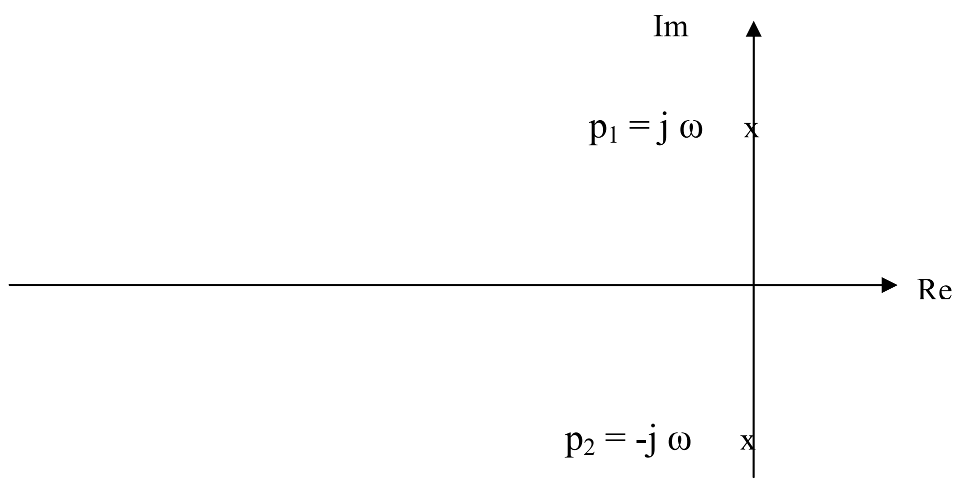
The poles of F(s) are given by

$s^2 + \omega^2 = 0$
or
$s^2 = -\omega^2$

that gives tow imaginary poles

$p_1 = j\,\omega$ and $p_2 = -j\,\omega$
and one non-zero pole p = - a .

s-plane representation is



Laplace Transform Table

See Table 2-1 Laplace Transform Pairs in the textbook.

Properties of Laplace Transform

Linearity

For $f(t)$ and g(t)with Laplace transforms

$F(s) = L\{f(t)\}$
$G(s) = L\{g(t)\}$

their linear combination

$a\, f(t) + b\, g(t)$

has the Laplace transform

$a\, F(s) + b\, G(s)$

Time translated function

A f(t) time translated by a time duration "a" is f(t-a) i.e.
f(t) for t=0 has the same value a the translated function f(t-a) at t=a

f(t) =0 for t<0, can be written as
f(t-a)1(t-a)
where unit step function translated by a is given by

1(t-a) =1 for t>a
0 for t<a

Given

$F(s) = L\{f(t)\}$

$L\{f(t - a)\} = e^{-as} F(s) \qquad \text{for } a \geq 0$

Example: Laplace Transform of Pulse Function f(t) of amplitude A and duration "a" is
f(t)=(A/a) 1(t) - (A/a) 1(t-a)
L{f(t)}= (A/as)(1- exp(- as))

Real Differentiation

Given F(s) = L{f(t)} and initial conditions of f(t), Lapalce transforms of derivatives of f(t) are obtained as follows

-first derivative of f(t)

$$L[\frac{d}{dt} f(t)] = sF(s) - f(0)$$

where $f(0)$ is the initial value of $f(t)$ evaluated at t = 0.
-second derivative of f(t)

$$L[\frac{d^2}{dt^2} f(t)] = s^2 F(s) - sf(0) - \dot{f}(0)$$

-n-th derivative of f(t)

$$L[\frac{d^n}{dt^n} f(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} \dot{f}(0) ..... - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

For zero initial values Laplace transform of the *n*th derivative of $f(t)$ is given by x^n F(s).
A time derivative in the time domain becomes a multiplication by *s* in the Laplace domain.

Example

Given that cos (ωt) =(1/ω) d/dt [sin (ωt)] and

$$L\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}$$

$$L\{\cos(\omega t)\} = L\{ d/dt [\sin(\omega t)] \} = s \frac{1}{\omega} \frac{\omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2}$$

Real Integration

$$\mathcal{L}\int_0^\infty f(t)dt = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$$

where $F(s) = \mathcal{L}\{f(t)\}$ and $f^{-1}(0) = \int f(t)\, dt$ evaluated at t=0.

Final Value Theorem

If F(s) has all poles on the left-hand side of the imaginary axis and no more than a single pole in the origin, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s \cdot F(s)$$

Steady state response of a system in the time domain can be obtained from the limit as s goes to zero of the Laplace transform of the function multiplied by s.

(See Example 2-2)

Initial Value Theorem

$$\lim_{t\to 0} f(t) = f(0) = \lim_{s\to\infty} s \cdot F(s)$$

<u>**Convolution Integral**</u>

If
$F(s) = L\{f(t)\}$
and
$G(s) = L\{g(t)\}$

then the inverse Laplace transform of their product

$H(s) = F(s) \cdot G(s)$

denoted $f(t)*g(t)$ and called the convolution of $f(t)$ and g(t) is

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\{F(s)\cdot G(s)\} = \int_0^t f(t-\tau)g(\tau)d\tau$$

<u>*(See Table 2-2 Properties of Laplace Transforms)*</u>

2.3 <u>Inverse Laplace Transform</u>

Inverse Laplace transform of the complex function F(s) results in the corresponding time function *f*(t)

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds \quad \text{for } t > 0$$

In practice, Laplace transform in not obtained using the above complex integration but by using Laplace Transform Pairs table either directly or by processing F(s) until it is transformed in parts found the table table.

The method for using indirectly the Laplace Transform Pairs table by processing F(s) until it is transformed in parts found the table is the partial fraction expansion method.

<u>Partial Fraction Expansion</u>

$$F(s) = \frac{B(s)}{A(s)} = \frac{\sum_{j=0}^{n} B_j s^j}{\sum_{i=0}^{n} A_i s^i}$$

In control systems analysis, F(s) is frequently occurs in the form of a ratio of polynomials, called also a rational function

where A(s) and B(s) are polynomials in s. In applications, the highest power of s in A(s) be greater or equal to the highest power of s in B(s). If not, the numerator B(s) is divided by the denominator A(s) in order to produce a polynomial in s plus a remainder as the numerator of a new rational function.

If F(s) is transformed in a sum of components

$$F(s) = F_1(s) + F_2(s) + ...F_n(s)$$

Such that the inverse Laplace transforms of $F_1(s)$, $F_2(s)$, …, $F_n(s)$ are available from the Laplace Transform Pairs table

$$L^{-1}\{F(s)\} = L^{-1}\{F_1(s)\} + L^{-1}\{F_2(s)\} + ... + L^{-1}\{F_n(s)\} = f_1(s) + f_2(s) + ...f_n(s)$$

Partial fraction expansion method is applied differently, depending the types of poles:

a) distinct poles
b) multiple poles
c) complex conjugate poles

a) **Rational Functions with Distinct Poles**

After the calculation of the roots of

$$A(s) = \sum_{j=1}^{n} a_j s^j = 0,$$

The zeros z_1, z_2,..,z_m and

$$B(s) = \sum_{i=1}^{n} b_i s^i = 0,$$

the poles p_1, p_2,..,p_n , where $n \geq m$,
the numerator and denominator polynomials can be factored as follows

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)(s+z_2)..(s+z_m)}{(s+p_1)(s+p_2)...(s+p_n)}$$

This form of the can be converted into a sum of simple partial fractions that can be found in the Laplace Transform Pairs table

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)(s+z_2)..(s+z_m)}{(s+p_1)(s+p_2)...(s+p_n)} = \frac{a_1}{(s+p_1)} + \frac{a_2}{(s+p_2)} + ... + \frac{a_n}{(s+p_n)}$$

What would be required is to find the constant a_k, called residues, for k=1,2,…,n, corresponding to the n-poles – p_k, such that the above right hand side sum of partial function is equal to F(s).
a_k is obtained as follows

$$\left[(s+p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} =$$

$$\left[(s+p_k)\frac{a_1}{(s+p_1)}+(s+p_k)\frac{a_2}{(s+p_2)}+...+(s+p_k)\frac{a_k}{(s+p_k)}+(s+p_k)\frac{a_n}{(s+p_n)}\right]_{s=-p_k} = a_k$$

i.e a_k is obtained from,

$$a_k = \left[(s+p_k)\frac{B(s)}{A(s)}\right]_{ss=-p_k}$$

b) **Functions with Multiple Poles**

$$F(s)=\frac{B(s)}{A(s)}=\frac{\sum_{j=0}^{m} B_j s^j}{(s+p_1)^{N_1}(s+p_2)^{N_2}...(s+p_n)^{N_n}}$$

has poles order of multiplicity higher or equal to 1, N_1 for p_1, N_2 for p_2, …., N_n for p_n,

$$F(s)=\frac{B(s)}{A(s)}=\frac{\sum_{j=0}^{m} B_j s^j}{(s+p_1)^{N_1}(s+p_2)^{N_2}...(s+p_n)^{N_n}}$$

$$=\frac{a_{N1}}{(s+p_1)^{N_1}}+\frac{a_{N_1-1}}{(s+p_1)^{N_1-1}}+...\frac{a_1}{(s+p_1)}+\frac{a_{N2}}{(s+p_2)^{N_2}}...+\frac{a_n}{(s+p_n)}$$

This can be transformed into a sum of simple partial fractions, which would require to find the constants a_{Nk}, corresponding to the n-poles – p_k, such that the above right hand side sum of partial function is equal to F(s).

a_{Nk} is obtained as shown in the example from the textbook, page 35-36 in

(Partial-Fraction Expansion when F(s) Involves Multiple Poles)

c) Complex conjugate poles

If p_1 and p_2 are complex conjugates poles, then the residues a_1 and a_2 are also complex conjugates such that only one needs to be evaluated.

From Laplace Transform Pairs table

$$L^{-1}\left[\frac{a_k}{(s+p_k)}\right]=a_k e^{-p_k t}$$

$$f(t)=L^{-1}[F(s)]=a_1 e^{-p_1 t}+a_2 e^{-p_2 t}+...+a_n e^{-p_n t} \qquad \text{for } t \geq 0$$

(See Examples 2-4 to 2-5, A-2-14)

Partial-Fraction Expansion with MATLAB

Consider the following function B(s)/A(s)

$$F(s) = \frac{B(s)}{A(s)} = \frac{\sum_{j=0}^{n} B_j s^j}{\sum_{i=0}^{n} A_i s^i} = \frac{num}{den}$$

MATLAB program

num= [B_n B_{n-1}…B_0]
den=[A_m A_{m-1}…A_0]

The command for calculating a_k

[r,p,k] = residue(num,den)

The residues a_k and the poles p_k, give the k-th a partial-fraction

$\frac{a_k}{(s+p_k)}$ for all k = 1, 2,...,n

(See Examples 2-6 and 2-7)

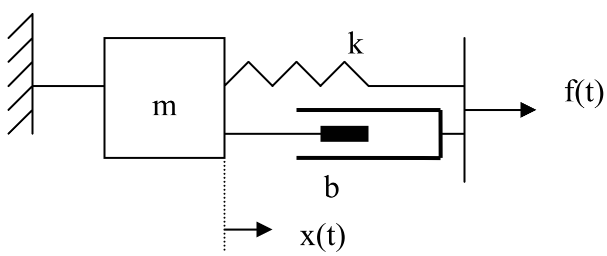
2.4 Solving Linear Time Invariant-Ordinary Differential Equations (LTI-ODE)

Laplace Transform LTI-ODE

(See Examples 2-8 and 2-9)

Example

a) This mass-damper-spring (m-b-k) system is subject to a applied force f(t) and has zero initial conditions. Calculate X(s).



Newton second law gives

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

Laplace Transform of this equation, for zero initial conditions, gives

$$ms^2X(s)+bsX(s)+kX(s)=F(s)$$

or

$(ms^2+bs+k)X(s)=F(s)$

Solving the above algebraic equation in ”s”

$X(s)=F(s) / (ms^2+bs+k)$

Inverse Laplace transform for unit impulse input force
$F(s)=1$
gives
$X(s)=1 / (ms^2+bs+k)$
or

$X(s)=(1 / m)/(s^2+sb/m+k/m)$

b) Obtain x(t) for b=0

$X(s)=(1 /m)/(s^2+ k/m)= (1/k)(k/m)/(s^2+ k/m)$
Laplace Transform Pairs table gives

$$L^{-1}\left[\frac{\omega}{(s^2+\omega^2)}\right]=\sin\omega t$$

Denote
$\omega^2=k/m$
such that X(s) can be written as follows
$X(s)=(1 /m)/(s^2+ k/m)= (1/k)(k/m)/(s^2+ k/m)=$
$(1/k)(\sqrt{k/m}) (\sqrt{k/m})/(s^2+ k/m)= (1/k)(\sqrt{k/m}) (\omega)/(s^2+ \omega^2)$
Consequently
$x(t)= (1/k)(\sqrt{k/m}) \sin \omega t= (1/\sqrt{km}) \sin (\sqrt{k/m})t$

(See Examples A-2-2 to A-2- 17)