<u>Chapter 3 MATHEMATICAL MODELING OF DYNAMIC SYSTEMS</u>

<u>3.1 System Modeling</u>

Mathematical Modeling

In designing control systems we must be able to model engineered system dynamics.

The model of a dynamic system is a set of equations (differential equations) that represents the dynamics of the system using physics laws.

The model permits to study system transients and steady state performance.

<u>Model complexity</u>

- As model becomes more detailed it also can become more accurate.

- Model accuracy needed for control system design is normally simpler than the model used for system simulation.

- Simpler models:
  -ignore some physical phenomena,
  -approximate linearly nonlinear characteristics
  -use lumped parameters approximation of distributed parameters systems.

- For the design of a control system:
  -an initial simplified model is built for conceptual design
  -a more model is used for controller design and parameters determination

Linear Systems

For linear systems the principle of superposition is valid, and the response to a complex input can be calculated by summing up the responses to its components.

Linear Time Invariant (LTI) Systems versus Linear Time Varying Systems

- Linear Time Invariant (LTI) Systems = systems:
  - represented by lumped components,-
  - described by linear differential equations
  - parameters of the equations are time invariant.

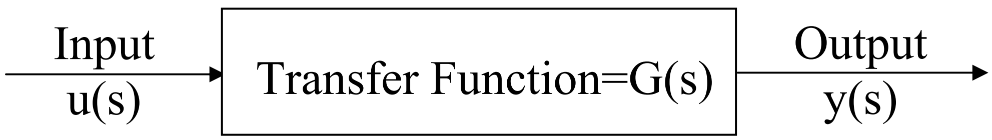
- Systems with parameters that vary in time are called linear time varying systems.

Examples: a car in motion or a rocket in flight have weight that diminishes as the fuel is consumed.

3.2 Transfer Function

$$\text{Transfer Function} = G(s) = \frac{\mathcal{L}\{\text{output}\}}{\mathcal{L}\{\text{input}\}}$$

for zero initial conditions.



$$G(s) = \frac{y(s)}{u(s)}$$

The transfer function of a system represents the link between the input to the system to the output of the system.

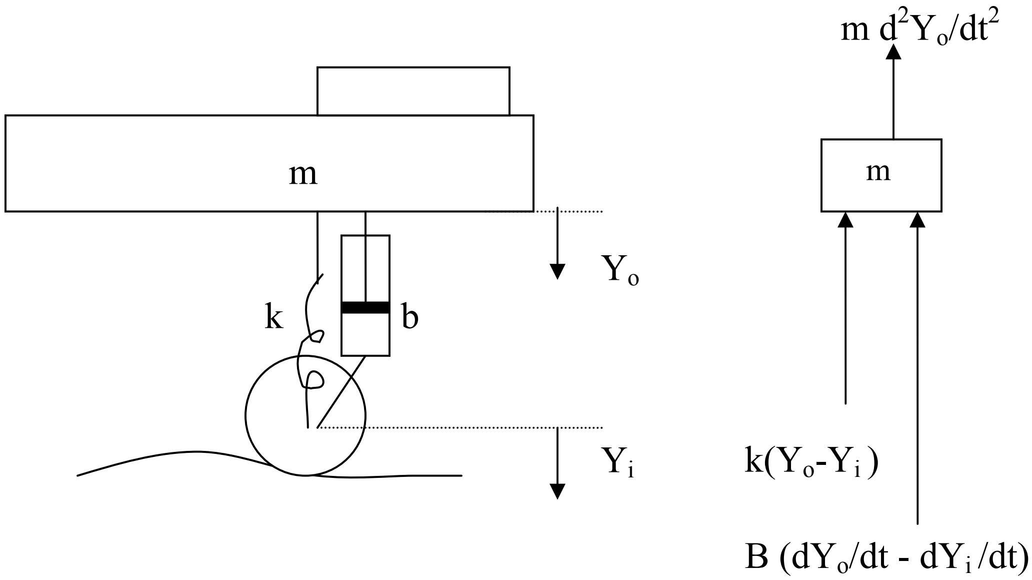
The transfer function of a system G(s) is a complex function that describes system dynamics in s-domains opposed t the differential equations that describe system dynamics in time domain.

The transfer function is independent of the input to the system and does not provide any information concerning the internal structure of the system. Same transfer function can represent different systems.

The transfer function permits to calculate the output or response for various inputs.

The transfer function can be calculated analytically starting from the physics equations or can be determined experimentally by measuring the output to various known inputs to the system.

Example: Car suspension model



$$m\frac{d^2Y_o}{dt^2} + b\frac{dY_o dt - dY_i dt}{dt} + k(Y_o - Y_i) = 0$$

Laplace transform for zero initial conditions gives

$$ms^2Y_o(s) + bs(Y_o(s) - Y_i(s)) + k(Y_o(s) - Y_i(s)) = 0$$

or

$$ms^2Y_o(s) + bsY_o(s) + kY_o(s) = bsY_i(s) + kY_i(s)$$

The transfer function is

$$\frac{Y_o(s)}{Y_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Impulse Response

The Laplace transform of an impulse function $\delta(t)$ is given by

$$\mathcal{L}\{\delta(t)\} = 1$$

The output of a system due to an impulse input $u(s)= \delta(s) = 1$ is

$$y(s) = G(s) \cdot u(s) = G(s)$$

The impulse response of a system is identical to the transfer function of that system.

The inverse Laplace transform of the impulse response G(s)

$$\mathcal{L}^{-1}\{G(s)\} = g(t)$$

The transfer function, that contains complete information about the dynamic characteristics of a system, can be obtained by applying an impulse input $u(t)= \delta(t)$ and measuring system response y(t) which in this case is identical to g(t). The transfer function will then be $G(s)=L\{g(t)\}$

3.3 Block Diagrams

Block diagrams are a graphical representation of the system model.

The blocks represent physical or functional components of the system.

Each block has inscribed the transfer function of that component the relate the output of the component to its input.

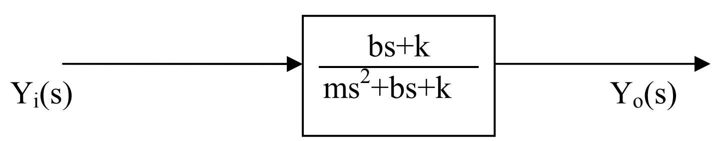
Block Diagrams

Block diagrams consist of

1. blocks
2. summation junctions
3. paths
4. branching points

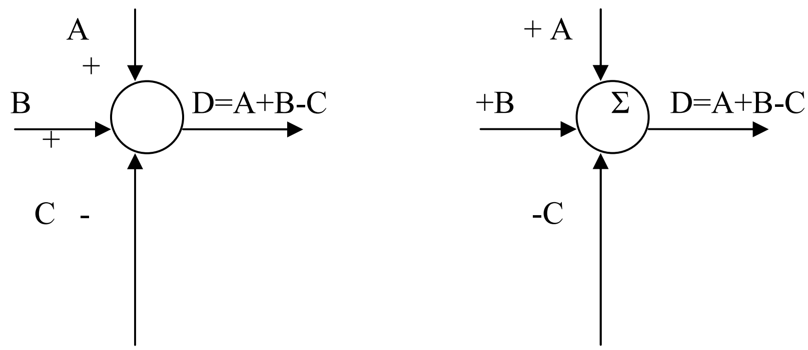
1. Block

Example: car suspension system



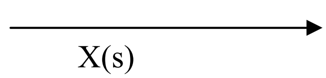
Blocks represent physical or functional components in the system. In the block is inscribed the transfer function of that component of the system

2. Summation Junction



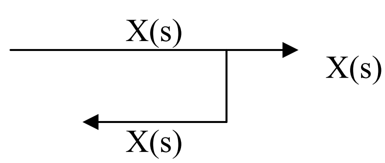
A summing junction results in the addition or subtraction of input signals for a single output.

3. Path



Signal X(s) flows along the directed path

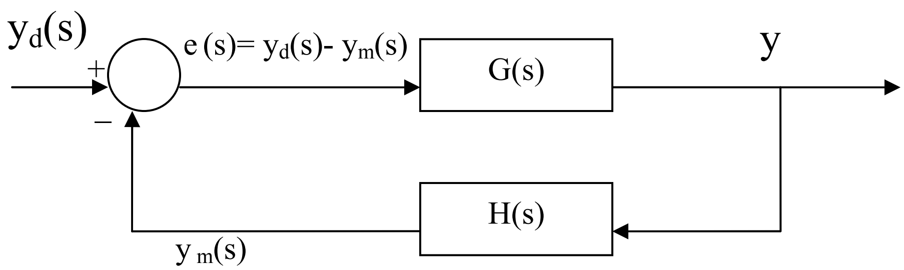
4. Branching Point



At the branching point a signal splits into two signals of the same value.

3.4 Block Diagram of a Closed Loop System

In a closed loop control system, also called feedback control system, the output variable y(s) is measured as $y_m(s)$, subtracted from its desired value $y_d(s)$ to calculate the error $e(s)= y_d(s) - y_m(s)$



G(s) is feed forward transfer function (of the controller, actuator and the system)

$$G(s) = y(s) / e(s)$$

H(s) is feedback transfer function (of the sensor)

$$H(s) = y_m(s) / y(s)$$

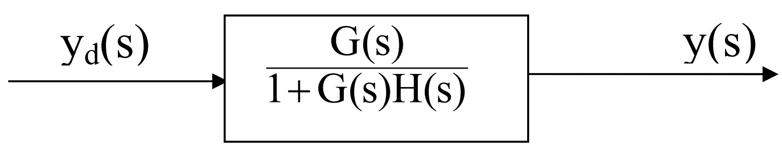
G(s) H(s) = open loop transfer function

$$G(s)\,H(s) = [y_m(s) / y(s)][\,y(s) / e(s)]= y_m(s) / e(s)$$

$y_d(s) / y(s)$ is closed loop transfer function obtained from the above equations by eliminating e(s) and $y_m(s)$

$$y(s) = G(s)\,e(s) = G(s)\,[y_d(s) - y_m(s)]= G(s)\,[y_d(s) - H(s)y\,(s)]$$
$$y(s) + G(s)\,H(s)\,y(s) =G(s)\,y_d(s)$$
$$y(s)/y_d(s)= G(s) / [1+ G(s)\,H(s)]$$

This equation gives the single block equivalent of the above closed loop system



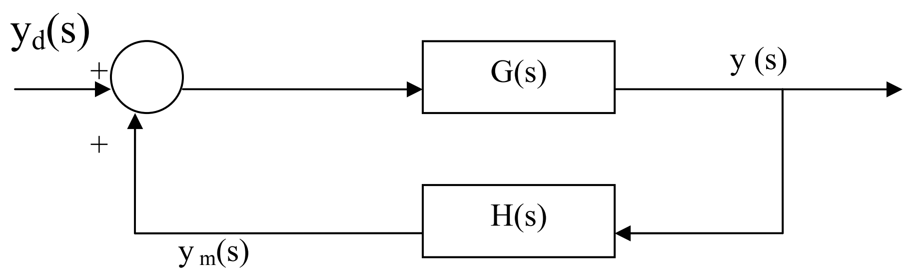
The transfer function represents the closed loop system dynamics with complex functions.

The output y(s) is given by

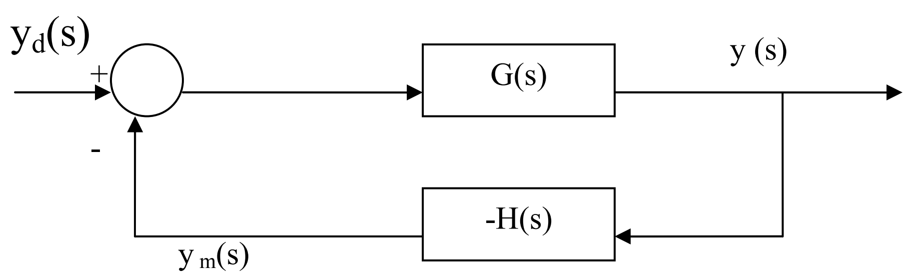
$$y(s)= G(s) / [1+ G(s)\ H(s)]\ y_d(s)$$

and depends on the closed loop transfer function and the desired value of the output $y_d(s)$, called also the input (to the closed loop system).

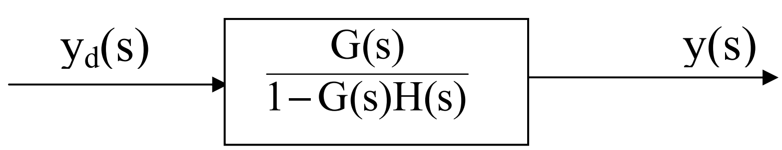
The following positive feedback block diagram



is equivalent to a negative feedback one if H(s) is replaced by –H(s)



which is equivalent to the block



Closed loop Control System Advantage

Ideally

$$y(s)/y_d(s) \rightarrow 1$$

i.e it is required that

$$G(s) / [1+ G(s) H(s)] \rightarrow 1$$

or

$$1 / [1/ G(s)+ H(s)] \rightarrow 1$$

which is achieved for a very high value of G(s)

$$G(s) \rightarrow \infty$$

G(s) represents the controller, actuator and the system and $G(s) \rightarrow \infty$ and the high feed forward gain can be achieved by a very high value of the controller transfer function. In the stability study it will be seen that there is a limit to such high value due to stability constraints.

Error

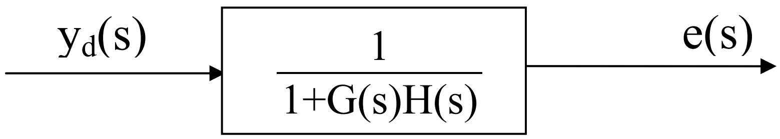
Error of a closed loop control system can be obtained from:

$e(s)=y_d(s)-y_m(s)= y_d(s)-H(s)\ y\ (s)=y_d(s)\{1- H(s)y(s)/y_d(s)\}=$

$y_d(s)\ \{1- H(s)G(s) / [1+ G(s)\ H(s)]\ \}=$

$\{1- H(s)G(s) / [1+ G(s)\ H(s)]\ \}y_d(s)=$

$\{1 / [1+ G(s)\ H(s)]\ \}y_d(s)$



Steady State Error

Final value theorem gives the steady state error of a system $e(\infty)$, i.e., error when $t \rightarrow \infty$

By taking the limit for $s\rightarrow 0$ of

$e(\infty)=\lim_{t\rightarrow\infty} e(t)= \lim_{s\rightarrow 0}\{se(s)\}= \lim_{s\rightarrow 0}\{1 / [1+ G(s)\ H(s)]\ \}y_d(s)=$

$\{1 / [1+ G(0)\ H(0)]\ \}y_d(0)$

The Components of Closed Loop Control System

-The system to be controlled S (s)

-The transducer (sensor) H(s) to measure the outputs y(s) and feed them back as $y_m(s)$

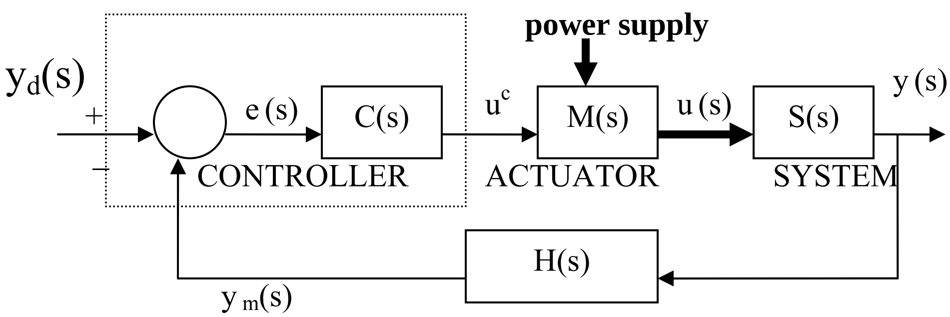
-the comparator to calculate the error e(s) between the desired value of the output $y_d(s)$ and output measurement $y_m(s)$

-the controller C(s) that uses this error signal e\(s) and generates the command u^c (s) to the system

-the actuator M(s)

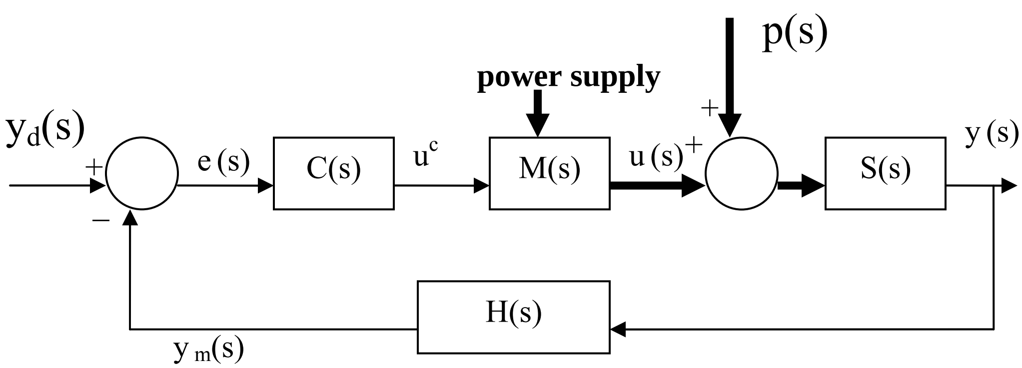
such that

$G(s) = C(s)\ M(s)\ S(s)$



Perturbations

Perturbations p (s) to a control system generally act on the system G(s), and occur as an additional power input that adds to the power input u(s) to the system.



Superposition principle is applied for linear systems and consider each input independently, and the outputs corresponding to each input alone can be added to give the total output.
The principle of superposition is applied as follows

-the output of the system y_1 (s) is calculated as a result of the input command $y_d(s)$ for p(s)=0.

$$y_1(s) = \frac{G(s)}{1+G(s)H(s)} y_d(s)$$

-the output of the system y_2 (s) is calculated as a result of the p(s) for $y_d(s)=0$.
$y_2 (s)=S(s)\{C(s)/[1+C(s)M(s)S(s)H(s)]\}p(s)$
or

$$y_2(s) = \frac{C(s)}{1+G(s)H(s)} p(s)$$

- add the two outputs $y(s)= y_1 (s)+ y_2(s)$

$$y(s) = y_1(s) + y_2(s) = \frac{G(s)}{1+G(s)H(s)} y_d(s) + \frac{C(s)}{1+G(s)H(s)} p(s) = \frac{G(s)y_d(s) + C(s)p(s)}{1+G(s)H(s)}$$

The effect of the perturbation p(s) is cancelled for $G(s)H(s)>>1$ and $M(s) >>1$.
For $G(s)H(s)>>1$
$1+G(s)H(s)\approx G(s)H(s)$
and
$C(s)/[1+G(s)H(s)]= C(s)/[1+C(s)M(s)S(s)H(s)] \approx C(s)/[C(s)M(s)S(s)H(s)] \approx 1/[M(s)S(s)H(s)]$
such that for

$$y(s) = y_1(s) + y_2(s) = \frac{1}{H(s)} y_d(s) + \frac{1}{M(s)S(s)H(s)} p(s)$$

For $M(s) >>1$, $1/[M(s)S(s)H(s)]\to 0$
and

$$y(s) = y_1(s) + y_2(s) = \frac{1}{H(s)} y_d(s)$$

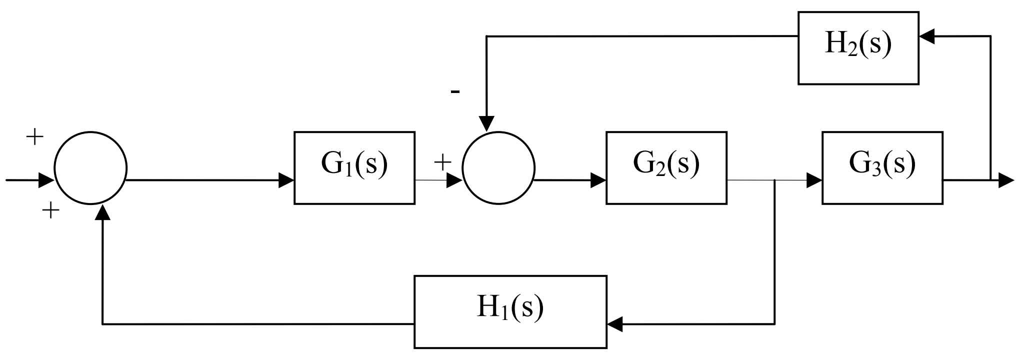
i.e. the output y(s) is not affected by p(s) and G(s).
H(s)=1 will result in the ideal
$y(s)= y_d(s)$

Block Diagram Reduction

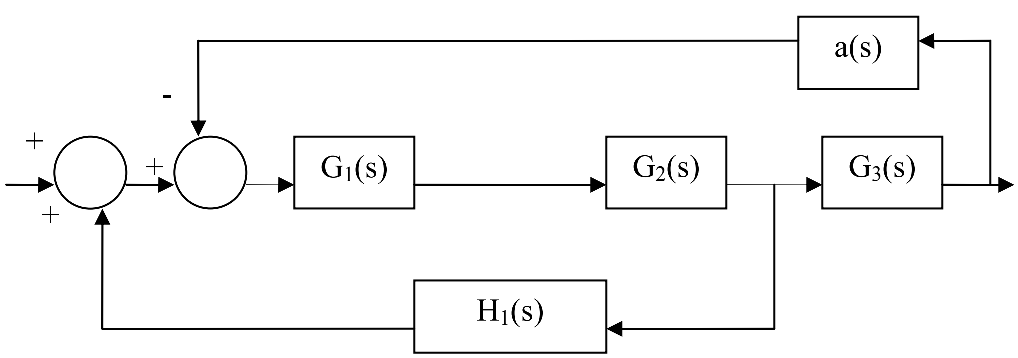
Block diagram for actual systems can contain a large number of blocks and block diagram reduction is required to reduce it to a single equivalent block, using the following two rules:

1. The product of the transfer functions in the feedforward direction should remain the same
2. The product of the transfer functions around the loop should remain the same

Example:



The top, negative feedback loop cannot be reduced unless the summing point is moved ahead $G_1(s)$.



We will determine the new, unknown, feedback transfer function a(s) using the above two rules.

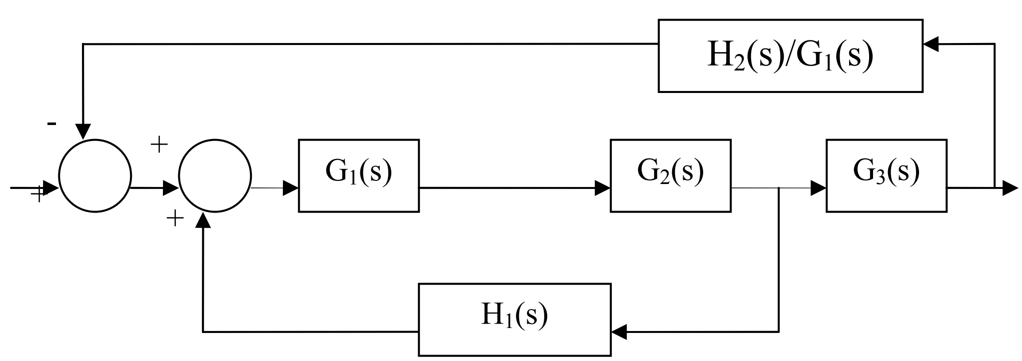
1. The first rule is actually satisfied, as both block diagrams have the same product of the transfer functions in the feedforward direction $G_1(s)\ G_2(s)\ G_3(s)$

2. The product of the transfer functions around the loop for the initial block diagram is

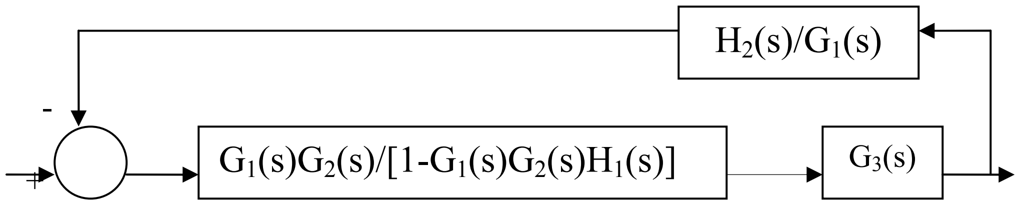
$G_1(s)\ [G_2(s)\ G_3(s)/(1+ G_2(s)G_3(s)\ H_2(s)]=$
$G_1(s)\ G_2(s)\ G_3(s)/[(1+ G_2(s)G_3(s)\ H_2(s)]$
and for the modified block diagram
$G_1(s)\ G_2(s)\ G_3(s)/(1+ G_1(s)G_2(s)G_3(s)\ a(s)]$

In order to be the same, given the same numerators, the denominators have to be equal
$1+ G_2(s)G_3(s)\ H_2(s)= 1+ G_1(s)G_2(s)G_3(s)\ a(s)$
or
$G_2(s)G_3(s)\ H_2(s)= G_1(s)G_2(s)G_3(s)\ a(s)$
or
$H_2(s)= G_1(s)\ a(s)$
which gives the solution for the unknown new feedback transfer function
$a(s) = H_2(s)/G_1(s)$

Previous block diagram is equivalent to



as the switch of the summing points satisfy the two rules.
Now the bottom positive feedback loop can be reduced to
$G_1(s)G_2(s)/[1-G_1(s)G_2(s)H_1(s)]$
such that



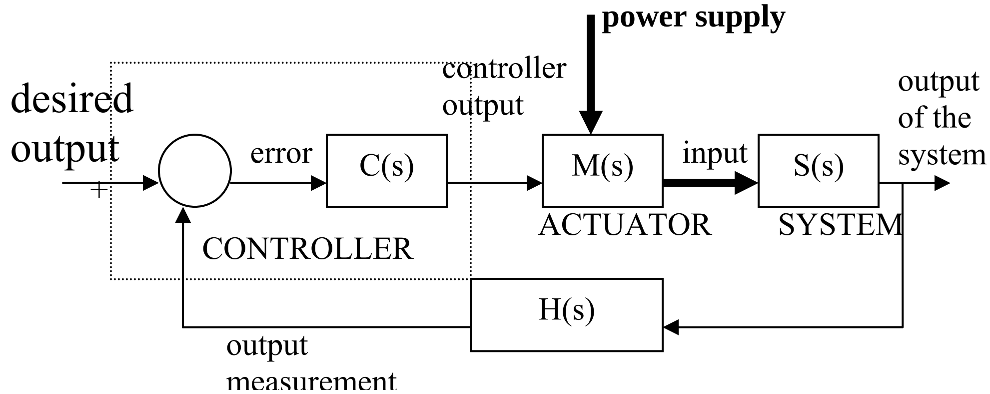
and, final reduction gives the final transfer function

$$G_1(s)G_2(s)\ G_3(s)/[1-G_1(s)G_2(s)H_1(s)]/\{1+ G_1(s)G_2(s)\ G_3(s)/[1-G_1(s)G_2(s)H_1(s)\ H_2(s)/G_1(s)]$$

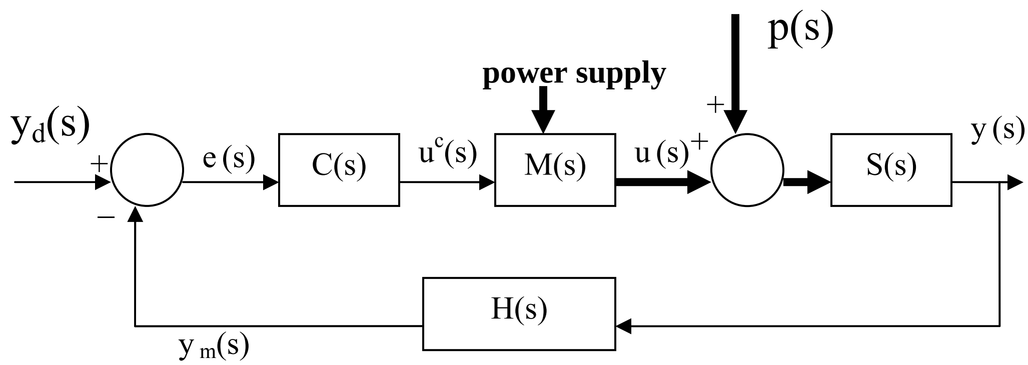
(See Example 3-2 and A-3-1 to A-3-5)

Controllers

Controllers are the part of the overall feedback control system that are the main focus of control engineering.



or



Classifications of Controllers

By the type of implementation:

-pneumatic controller

-hydraulic controller

-analog electronic controllers

-digital controllers

Modern implementation is as digital controllers. By control law:

> On-off,
> Proportional, P-control
> PD-control
> PID-control

In Control I and II only the last three are studied.

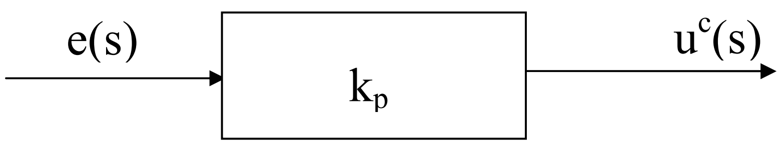
P-control

The transfer function is

$$\frac{u^c(s)}{e(s)} = k_p$$

where k_p is the proportional gain

The block diagram is



The P-controller corresponds to an operational amplifier with an adjustable gain k_p.

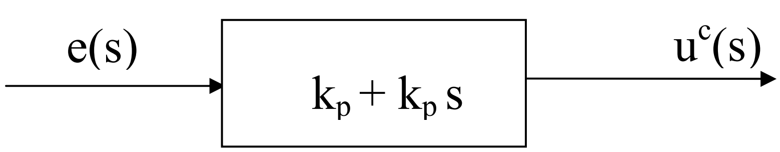
PD Control

The transfer function is

$$\frac{u^c(s)}{e(s)} = k_p + k_d s = k_p(1 + T_d s)$$

where k_d is the derivative gain and $T_d = k_d / k_p$

The block diagram is



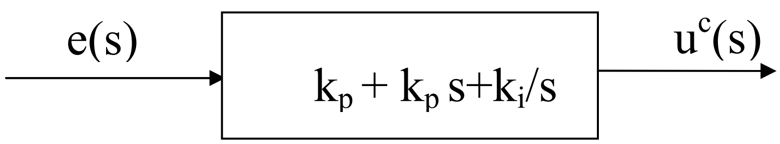
PID Control

The PDI controller transfer function of is

$$\frac{u^c(s)}{e(s)} = k_p + k_d s + \frac{k_i}{s} = k_p\left(1 + T_d s + \frac{1}{T_i s}\right)$$

where k_i is the integral gain and $1/T_i = k_i / k_p$

The block diagram is



PID controllers are frequently used in applications.

3.5 State Space Representation

State space based control approaches are developed in Russia in 1960s.

This is called Modern Control Theory or State space Control Theory or Time Domain Approach (as opposed to Conventional Control Theory of Classic Control Theory or Frequency Domain Approach

The major limitation of the conventional frequency domain models is the requirement that the system be linear and Single Input Single Output (SISO).

Definitions

State Variables: are the minimum set of variables that uniquely define the state of a dynamic system at any instant of time.

States: are the values of the set of state variables. For known current state values and of the input the state of the system in the future time can be calculated.

State Vector: is the vector of state variables.

State space: is the hyperspace in which the state of the system takes values.

State Space Model

consists of:

a) State equations for a nonlinear system

$$\dot{x}_1 = f_1(x_1, x_2, ..., x_n, u_1, u_2, ..., u_k, t)$$
$$\dot{x}_2 = f_2(x_1, x_2, ..., x_n, u_1, u_2, ..., u_k, t)$$
....
$$\dot{x}_n = f_n(x_1, x_2, ..., x_n, u_1, u_2, ..., u_k, t)$$

b) output equations for a nonlinear system

$$y_1 = g_1(x_1, x_2, ..., x_n, u_1, u_2, ..., u_k, t)$$
$$y_2 = g_2(x_1, x_2, ..., x_n, u_1, u_2, ..., u_k, t)$$
....
$$y_m = g_m(x_1, x_2, ..., x_n, u_1, u_2, ..., u_k, t)$$

Matrix state space model is defined for

-state vector [n·1]

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ x_n(t) \end{bmatrix}$$

-output vector [m·1]

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ . \\ . \\ y_m(t) \end{bmatrix}$$

-input vector [k·1]

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ . \\ . \\ u_k(t) \end{bmatrix}$$

- right hand side of state equation

$$f(x,u,t) = \begin{bmatrix} f_1(x_1, x_2, ..., x_n, u_{1,} u_2, ..., u_k, t) \\ f_2(x_1, x_2, ..., x_n, u_{1,} u_2, ..., u_k, t) \\ . \\ . \\ f_n(x_1, x_2, ..., x_n, u_{1,} u_2, ..., u_k, t) \end{bmatrix}$$

-right hand side of output equation

$$g(x,u,t) = \begin{bmatrix} f_1(x_1, x_2, ..., x_n, u_{1,} u_2, ..., u_k, t) \\ f_2(x_1, x_2, ..., x_n, u_{1,} u_2, ..., u_k, t) \\ . \\ . \\ f_n(x_1, x_2, ..., x_n, u_{1,} u_2, ..., u_k, t) \end{bmatrix}$$

the above state and output equations can be written in a compact form as matrix equations

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$
$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, t)$$

Linearization of Nonlinear Equations

The operation of a system for a short time duration is around an operating point, with small variations about the equilibrium point.
In this case, is suitable to model the operation by approximations of the nonlinear system using local linear approximations.
Such a linearized LTI model is used for controllers design.

Linearizing state and output nonlinear equations about a given value of the state vector, gives Linear Time Variant equations

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t)$$

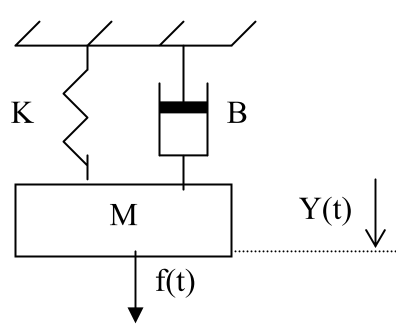
A(t) is state matrix
B(t) is the input matrix
C(t) is the output matrix
D(t) is the direct transmission matrix.

A Linear Time Invariant (LTI) system is given by

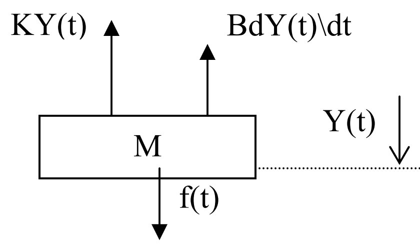
$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t)$$

where A, B, C, and D are matrices of constant values.

Example: A mass-damper-spring vertical M-B-K system is subject to a force f(t)



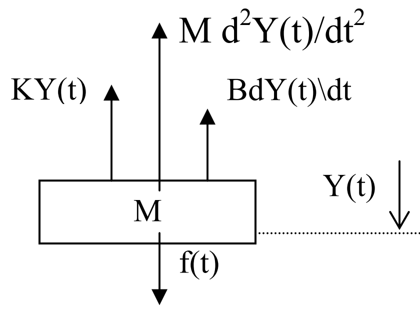
The free body diagram is



The equation of motion is given by Newton second law

$$M\frac{d^2Y(t)}{dt^2} = -B\frac{dY(t)}{dt} - KY(t) + f(t)$$

Inertia force is M $d^2Y(t)/dt^2$ and gives the following free body diagram



D'Alembert principle gives the force balance equation

$$M\frac{d^2Y(t)}{dt^2} + B\frac{dY(t)}{dt} + KY(t) - f(t) = 0$$

Let us define the following set of state variables $x_1(t)$ and $x_2(t)$
$x_1(t)=Y(t)$
$x_2(t)=dY(t)/dt= dx_1(t)/dt$
and
$dx_2(t) /dt= d^2Y(t)/dt^2 =(1/M)[-B\ dY(t)/dt-KY(t)+f(t)]$
The above second order equation of motion can be replaced by two first order state equations using state variables $x_1(t)$ and $x_2(t)$ and input variable $u(t)=f(t)$

$dx_1(t)/dt= x_2(t)$
$dx_2(t) /dt= -(K/M)\ x_1(t) - (B/M)\ x_2(t) + (1/M)\ u(t)]$

and system output equation
$y(t) =x_1(t)$

in case that the output is the position (or $y(t) =x_2(t)$ if the output is the velocity $x_2(t)=dY(t)/dt$ of the mass M)
Matrix form of state dynamics and output equations is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

or

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t)$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1/M \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$D=0$

Relationship between Transfer Functions and State-Space Equations

System transfer function

$$G(s) = \frac{y(s)}{u(s)}$$

has to be obtained from the matrix form of state and output equations by elimination the other variable, the vector x(s).
This requires Laplace transform of these matrix equations for zero initial conditions

$$s\mathbf{x}(s) = A\mathbf{x}(s) + B\mathbf{u}(s)$$
$$\mathbf{y}(s) = C\mathbf{x}(s) + D\mathbf{u}(s)$$

or

$$(sI - A)\mathbf{x}(s) = B\mathbf{u}(s)$$
$$\mathbf{y}(s) = C\mathbf{x}(s) + D\mathbf{u}(s)$$

where I is a 2 by 2 identity matrix.

The elimination is carried out by solving first equation for x(s)

$$\mathbf{x}(s) = (sI - A)^{-1}B\mathbf{u}(s)$$

and replacing x(s) in the second equation

$$\mathbf{y}(s) = C(sI - A)^{-1}B\mathbf{u}(s) + D\mathbf{u}(s)$$

or

$$\mathbf{y}(s) = [C(sI - A)^{-1}B + D]\mathbf{u}(s)$$

For single input , $\mathbf{y}(s) = y(s)$ and single output $\mathbf{u}(s) = u(s)$
such that

$$y(s) = [C(sI - A)^{-1}B + D]u(s)$$

This equation gives the relationship between the transfer function and the matrices A, B, C, and D of the state space representation

$$G(s)=\frac{y(s)}{u(s)}=[C(sI-A)^{-1}B+D]$$

Given that the inverse of the matrix s**I-A** is given by the matrix of adjoints ||s**I-A||** divided by the determinant |s**I-A|**

(s**I-A)$^{-1}$=** ||s**I-A||**/|s**I-A|**

Assuming D=0

G(s) =C||s**I-A||B/**|s**I-A|**

Such that the poles of the transfer function G(s) are given by

|s**I-A|=0**

that is actually the equation that permits the calculation of the vector of eigenvalues λ **of the square matrix A**

|λ **I-A|=0**

This indicates that the characteristic equation of the system is given by (s**I-A**). That means that the eigenvalues of **A** is identical to the poles of G(s) in the Laplace domain.

The poles of the transfer function G(s) can, consequently be calculated as eigenvalues of the matrix **A.**

Example: Consider the M-B-K system analysed before.

State space and output equations for the system were given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u(t)$$

$$y(t)=\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

To obtain the transfer function from the state space equations see

(See Example 3-4 from Ogata textbook)

The result is

G(s)= 1/(Ms^2+bs+k)

The poles of G(s) are given by

Ms^2+Bs+K=0

Which is the determinant of the matrix **A**,

|**A**|=0

<u>State Space representation of *n*th-order systems of Linear Differential Equations</u>

a) **The case when the forcing function does not involve derivative terms**

b) **When the input involves derivative terms in the forcing function**

(See Ch. 3-5 from Ogata textbook)

(See Example 3-5 in Ogata textbook)