Kinematic Analysis

A two-dimensional mechanical system that consists of both translational and rotational portions is the hanging crane shown in Fig. 2.9. The free-body diagrams are shown in Fig. 2.10. In the case of the pendulum, the forces are shown with bold lines while the components of the inertial acceleration of its center of mass are shown with dashed lines. The inertial acceleration needs to be determined because the \( \dot{a} \) in Eq. (2.1) is with respect to an inertial reference. The total inertial acceleration of the pendulum's mass center is the vector sum of the three dashed arrows shown. The derivation of the components of an object's acceleration, called \textit{kinematics}, is usually studied as a prelude to the application of Newton's Laws. Alternatively, one could express the center of mass of the pendulum as a vector from an inertial reference and then differentiate that vector twice to obtain an inertial acceleration.

Using the results of a kinematic study, we find that the component of acceleration along the pendulum is \( l \dot{\theta}^2 \) and is called the "centripetal" acceleration. It is present for any object whose velocity is changing direction. The \( \dot{x} \) component of acceleration is a consequence of the pendulum pivot point accelerating at the trolley's acceleration and will always have the same direction and magnitude as the trolley. The \( \dot{\theta} \) component is a result of angular acceleration of the pendulum and is always perpendicular to the pendulum. The total inertial acceleration will automatically be the vector sum of the three acceleration components shown in Fig. 2.10(b).

In order to understand the \( l \dot{\theta} \) and \( l \dot{\theta}^2 \) terms better, consider the situation in Fig. 2.10(c), where the \( \hat{i} \) and \( \hat{j} \) axes are inertially fixed. A vector \( \mathbf{r} \) describing the position of the pendulum center of mass can be expressed as

\[
\mathbf{r} = l(\sin \theta \hat{i} - \cos \theta \hat{j})
\]

The first derivative of \( \mathbf{r} \) is

\[
\dot{\mathbf{r}} = l \dot{\theta}(\cos \theta \hat{i} + \sin \theta \hat{j})
\]

Likewise, the second derivative of \( \mathbf{r} \) is

\[
\ddot{\mathbf{r}} = l \ddot{\theta}(\cos \theta \hat{i} + \sin \theta \hat{j}) - l \dot{\theta}^2(\sin \theta \hat{i} - \cos \theta \hat{j})
\]

Note that the \( l \dot{\theta}^2 \) term is aligned along the pendulum pointing toward the axis of rotation and that the \( l \dot{\theta} \) term is aligned perpendicular to the pendulum pointing in the direction of a positive rotation.

The total inertial acceleration \( \ddot{\mathbf{r}} \) can be expressed as

\[
\ddot{\mathbf{r}} = \ddot{\mathbf{a}}_{\text{CM}} + \ddot{\mathbf{a}}_{\text{ext}} = \ddot{\mathbf{a}}_{\text{CM}} + \frac{\ddot{\mathbf{f}}_{\text{CM}}}{m_{\text{cm}}} + \frac{\ddot{\mathbf{f}}_{\text{ext}}}{m_{\text{ext}}}
\]

and

\[
\ddot{\mathbf{a}}_{\text{CM}} = \ddot{\mathbf{a}}_{\text{ext}} + m \ddot{\mathbf{a}}_{\text{CM}} = \ddot{x} \hat{i} + \ddot{y} \hat{j}
\]

The absolute acceleration of the pendulum in \( \hat{i} \hat{j} \) coordinate is

\[
\ddot{a}_{\text{CM}} = \left( \ddot{x} \hat{i} + l \ddot{\theta} \cos \theta \hat{i} - l \ddot{\theta}^2 \sin \theta \hat{j} \right) \hat{i} + \left( l \ddot{\theta} \sin \theta - l \dot{\theta}^2 \cos \theta \right) \hat{j}
\]

\[
\ddot{\mathbf{a}}_{\text{CM}} = \frac{\ddot{x} \sin \theta + l \ddot{\theta}^2}{m_{\text{cm}}} \left( \ddot{x} \hat{i} + \ddot{y} \hat{j} \right) \hat{i}
\]
Dynamics Analysis

Having all the forces and accelerations for the two bodies, we proceed to apply Eq. (2.1). In the case of the trolley Fig. 2.10(a), we see that it is constrained by the tracks to move only in the \( x \) direction; therefore, application of Eq. (2.1) in this direction yields

\[ \Sigma F_x = 0 \quad m_i \ddot{x} + b \dot{x} = u - N. \]  

(2.8)

Conceptually, Eq. (2.1) can be applied to the pendulum of Fig. 2.10(b) in the vertical and horizontal directions, and Eq. (2.5) can be applied for rotational motion to yield three equations in the three unknowns: \( N, P, \) and \( \theta \). These three equations can then be manipulated to eliminate the reaction forces \( N \) and \( P \) so that a single equation results describing the motion of the pendulum, i.e., a single equation in \( \theta \). For example, application of Eq. (2.1) in the \( x \) direction yields

\[ \Sigma F_x = 0 \quad N = m_i \ddot{x} + m_p l \dot{\theta} \cos \theta - m_p l \dot{\theta}^2 \sin \theta. \]  

(2.9)

However, considerable algebra will be eliminated if Eq. (2.1) is applied perpendicular to the pendulum to yield

\[ \Sigma F = 0 \quad \begin{align*} P \sin \theta + N \cos \theta - m_p g \sin \theta &= m_p l \dot{\theta} + m_i l \dot{\theta} \cos \theta. \end{align*} \]  

(2.10)

Application of Eq. (2.5) for the rotational pendulum motion where the moments are summed about the center of mass yields (i.e., no torques present)

\[ \Sigma M_{CM} = 0 \quad (-P l \sin \theta - N l \cos \theta) - I \ddot{\theta}, \]  

(2.11)

where \( I \) is the moment of inertia about the pendulum's mass center. The reaction forces, \( N \) and \( P \), can now be eliminated relatively easily by combining Eqs. (2.10) and (2.11). This yields the equation

\[ \left( I + m_i l^2 \right) \ddot{\theta} + m_p gl \sin \theta = -m_p l \dot{\theta} \cos \theta. \]  

(2.12)

It is identical to a simple pendulum equation of motion, except that it contains a forcing function that is proportional to the trolley's acceleration.

An equation describing the trolley motion was found in Eq. (2.8), but it contains the unknown reaction force, \( \dot{N} \). \( N \) can be eliminated by combining Eqs. (2.9) and (2.8). This yields

\[ (m_i + m_p) \ddot{x} + b \dot{x} + m_p l \ddot{\theta} \cos \theta - m_p l \dot{\theta}^2 \sin \theta = u. \]  

(2.13)

Equations (2.12) and (2.13) constitute the differential equations that describe the motion of the crane with its hanging load. For an accurate calculation of the motion of the system, these nonlinear equations need to be solved. However, nonlinear equations are much more difficult to solve than linear ones and the kinds of possible motions resulting from a nonlinear model are much more difficult to categorize than those resulting from a linear model. It is therefore useful to linearize models in order to gain access to linear analysis methods. It may be that the linear models and linear analysis are used only for the design of the control system (whose function may be to maintain the system in the linear region). Once a control system is synthesized and shown to have desirable performance based on linear analysis, it is then prudent to carry out an accurate analysis of the system with all the nonlinearities in order to validate that performance.

To linearize the equations of motion of the hanging crane, we could assume the pendulum has small motions about the vertical where \( \cos \theta \approx 1, \sin \theta \approx \theta, \) and \( \theta^2 \approx 0; \) thus, the equations may be approximated by

\[ \begin{align*} (I + m_i l^2) \ddot{\theta} + m_p gl \theta &= -m_p l \dot{x} \\ (m_i + m_p) \ddot{x} + b \dot{x} + m_p l \ddot{\theta} &= u. \end{align*} \]  

(2.14)

\[ \text{Control problem: Find } u(t) \text{ such that } x(t) \text{ is given} \]  

\[ -\frac{\pi}{2} < \theta(t) < \frac{\pi}{2} \]  

\[ -\frac{\pi}{8} < \dot{\theta}(t) < \frac{\pi}{8} \]